Solutions of Negative Pell Equation Involving Cousin Primes

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Abstract - **We illustrate recent development in computational number theory by studying their implications in solving the Pell's equation. In this paper, we search for finding non – trivial integral solutions to the Pell's equation $x^2 = 17y^2 - 13^t$ for all choices of $t \in N$. Recurrence relations among the solutions are also obtained**.

Keywords - **Pell's equation, Integer solution, Diophantine equation, Cousin Prime, Emirp.**

I. INTRODUCTION

The Pell's equation is the equation $x^2 = dy^2 \pm N$ to be solved in positive integer x, y for a non – zero integer d. For example, for $d = 5$ one can take $x = 9, y = 4$. We shall always assume that d is positive but not a square, since otherwise there are clearly no solutions. Pell's equation has an extra ordinarily rich history to which Weil [7] is the best guide. A particularly lucid exposition of method of solving the Pell equation is found in Euler's algebra [15 &16].

Cousin primes are prime numbers that differ by four. Here using two consecutive cousin primes 13 & 17 we form a Pell's equation $x^2 = 17y^2 - 13^t$, $t \in \boldsymbol{N}$ and search for its non- trivial integer solutions. In addition, 13 &17 are *emirp* also.

This communication concerns with the Pell equation $x^2 = 17y^2 - 13^t, t \in \boldsymbol{N}$, and infinitely many positive integer solutions are obtained for the choices of t given by $(i)\ t = 1$, $(ii)\ t = 3$ $(iii) t = 5$ $(iv)\ 2k$ and $t = 2k + 5$. Further recurrence relations on the solutions are derived.

II. PRELIMINARIES

Testing the solubility of the negative Pell equation:

Suppose $D$is a positive integer, not a perfect square. Then the negative Pell equation $x^2 - Dy^2 = -1$ is soluble if and only if $D$is expressible as $D = a^2 + b^2$, $\gcd(a, b) = 1$. If a and b positive, b odd and the Diophantine equation $-bV^2 + 2aVW + bW^2 = 1$ has a solution (The case of solubility occurs for exactly one each (a, b)).

The algorithm:

1. Find all expression of D sum of two relatively prime square using *cornacchia's* method. If none, exit – the negative Pell equation is not soluble.
2. For each representation $D = a^2 + b^2$, $\gcd(a, b) = 1$, $a$and b positive, b odd, test the solubility of $-bV^2 + 2aVW + bW^2 = 1$ using the Lagrange – Matthews algorithm. If soluble, exit – the negative Pell equation is soluble.
3. If each representation yields no solution, then the negative Pell equation is insoluble.

Proposition 1:

Let p be a prime. The negative Pell's equation

$$x^2 - py^2 = -1$$

is solvable if and only if $p = 2$ or $p \equiv 1(mod\ 4)$.
This paper concerns with a negative Pell equation

$$x^2 = 17y^2 - 13^t,\ \ t \in \boldsymbol{N}$$

Here we consider the prime 17 which confirms the existence of integer solutions of using Proposition 1.

III. METHOD OF ANALYSIS

For this Particular equation we consider the prime $p = 17$. Since p satisfies all the conditions of Proposition 1, we can conclude that the negative Pell equation $x^2 = 17y^2 - 13^t, t \in \boldsymbol{N}$ is solvable in integers.

(OR) By Testing the solubility of the negative Pell equation solving $a^2 + b^2 = 17$. Here $(a, b) = (4, 1)$.

Number of positive primitive solutions with $a \geq b$is one.

1. Testing $(a, b) = (4,1)$
2. $-V^2 + 8VW + W^2 = 1$ has a solution $(V, W) = (8, 1)$.
   So $x^2 = 17y^2 - 13^t$ is solvable.

A. Choice 1: $t = 1$

The Pell equation is

$$x^2 = 17y^2 - 13 \tag{1}$$

Let (x_0, y_0) be the initial solution of (1) given by

$$x_0 = 2\ ;\ \ y_0 = 1$$

To find the other solutions of (1), consider the Pell equation

$$x^2 = 17y^2 + 1$$

whose initial solution $(\widetilde{x_n},\ \widetilde{y_n})$ is given by

$$\widetilde{x_n} = \frac{1}{2} f_n$$

$$\widetilde{y_n} = \frac{1}{2\sqrt{17}}\ g_n$$

where $f_n = \left(33 + 8\sqrt{17}\right)^{n+1} + \left(33 - 8\sqrt{17}\right)^{n+1}$

$g_n = \left(33 + 8\sqrt{17}\right)^{n+1} - \left(33 - 8\sqrt{17}\right)^{n+1}, \quad n = 0,1,2, \cdots$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n},\ \widetilde{y_n})$, the sequence of non – zero distinct integer solution to (1) are obtained as

$$x_{n+1} = \tfrac{1}{2}\left[2f_n + \sqrt{17}\, g_n\right] \tag{2}$$

$$y_{n+1} = \tfrac{1}{2\sqrt{17}}\left[\sqrt{17}\, f_n + 2g_n\right] \tag{3}$$

The recurrence relation satisfied by the solutions of (1) are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0$$

$$y_{n+2} - 66y_{n+1} + y_n = 0$$

B. Choices 2: $t = 3$

The Pell equation is

$$x^2 = 17y^2 - 13^3 \tag{4}$$

Let (x_0, y_0) be the initial solution of (4) given by

$$x_0 = 110;\ \ \ y_0 = 29$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solution to (4) are obtained as

$$x_{n+1} = \frac{1}{2}\left[110\, f_n + 29\sqrt{17}\, g_n\right] \quad (5)$$

$$y_{n+1} = \frac{1}{2\sqrt{17}}\left[29\sqrt{17}\, f_n + 110\, g_n\right] \quad (6)$$

The recurrence relations satisfied by the solutions of (4) are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0$$

$$y_{n+2} - 66y_{n+1} + y_n = 0$$

C. Choices 3: $t = 5$

The Pell equation is

$$x^2 = 17y^2 - 13^5 \quad (7)$$

Let (x_0, y_0) be the initial solution of (7) given by

$$x_0 = 1358; \quad y_0 = 361$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solution to (7) are obtained as

$$x_{n+1} = \frac{1}{2}\left[1358\, f_n + 361\sqrt{17}\, g_n\right] \quad (8)$$

$$y_{n+1} = \frac{1}{2\sqrt{17}}\left[361\sqrt{17}\, f_n + 1358\, g_n\right] \quad (9)$$

The recurrence relations satisfied by the solutions of (7) are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0$$

$$y_{n+2} - 66y_{n+1} + y_n = 0$$

D. Choices 4: $t = 2k, k \in N$

The Pell equation is

$$x^2 = 17y^2 - 13^{2k}, \; k \in \boldsymbol{N} \quad (10)$$

Let (x_0, y_0) be the initial solution of (10) given by

$$x_0 = 13^k.4\,; \quad y_0 = 13^k$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solution to (10) are obtained as

$$x_{n+1} = \frac{13^k}{2}\left[4\, f_n + \sqrt{17}\, g_n\right] \quad (11)$$

$$y_{n+1} = \frac{13^k}{2\sqrt{17}}\left[\sqrt{17}\, f_n + 4\, g_n\right] \quad (12)$$

The recurrence relations satisfied by the solutions of (10) are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0$$

$$y_{n+2} - 66y_{n+1} + y_n = 0$$

E. Choices 5: $t = 2k + 5, \; k \in N$

The Pell equation is

$$x^2 = 17y^2 - 13^{2k+5}, \; k \in \boldsymbol{N} \quad (13)$$

Let (x_0, y_0) be the initial solution of (13) given by

$$x_0 = (3970)13^{k-1}\,; \quad y_0 = (2149)13^{k-1}$$

Applying Brahma Gupta lemma between (x_0, y_0) and $(\widetilde{x_n}, \widetilde{y_n})$, the sequence of non – zero distinct integer solution to (13) are obtained as

$$x_{n+1} = \frac{13^{k-1}}{2}\left[3970\, f_n + 2149\sqrt{17}\, g_n\right] \quad (14)$$

$$y_{n+1} = \frac{13^{k-1}}{2\sqrt{17}}\left[2149\sqrt{17}\, f_n + 3970\, g_n\right] \quad (15)$$

The recurrence relations satisfied by the solutions of (13) are given by

$$x_{n+2} - 66x_{n+1} + x_n = 0$$
$$y_{n+2} - 66y_{n+1} + y_n = 0$$

IV. CONCLUSION

Solving a Pell's equation using the above method provides powerful tool for finding solutions of equations of similar type. Neglecting any time consideration it is possible using current methods to determine the solvability of Pell – like equation.

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