# Understood the Significance of Euler Formula And Study of Vector Spaces

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*Abstract* - **In this paper study of Euler's numerous for contributions and his formula. The number of vertices (V), the number of edges (E), and the number of faces (F), of a convex polyhera in an alternating sum V−E+F = 2 is most well known and easily understand. The Euler significance (Characteristic) χ is then defined as χ = V −E +F. Using the Euler characteristic of the graph theory, algebraic topology and then use the tools to give an overview. This focus of a cycle basis is a set of cycles that generates the cycle space. In the mathematical discipline of graph theory, the edge space and vertex space of an undirected graph are vector spaces defined in terms of the edge and vertex sets, respectively. These vector spaces make it possible to use techniques of linear algebra in studying the graph.**

*Keywords* **---Euler's Formula; Polyhera; Euler significance; algebraic topology; vector spaces; vertex space**

## I.INTRODUCTION

We can interpret the sad problem as a problem about graphs. Given sets A1,A2,…,An with $\bigcup_{i=1}^{n}$Ai={x1,x2,…,xm} we define a graph with n+m vertices as follows: The vertices are labeled {A1,A2,…,An,x1,x2,…xm} and the edges are{{Ai,xj}|xj∈Ai}.

Euler's Formula for graphs, and then suggest why it is true for polyhedral. (Don't panic if you don't know what Euler's Formula is; all will be revealed shortly!) If you haven't met the idea of a graph before (or even if you have!), you might like to have a look here . I am also going to assume for the main proof that you are familiar with the idea of induction, although you may still be able to get the idea of the article without being.

As explained in the article referred to above, a graph is a mathematical object consisting of points (vertices) and lines (edges) joining some or all of the pairs of vertices. The lines may be curved, and may overlap, but may only intersect at vertices.

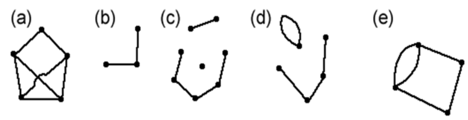


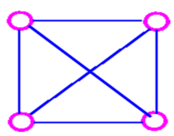

**Figure 1:** first picture

*A. Graph Planarity*

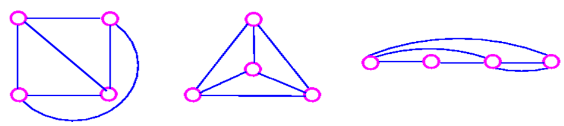
A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident. Any such drawing is called a plane drawing of G.

For example, the graph $K_4$ is planar, since it can be drawn in the plane without edges crossing.

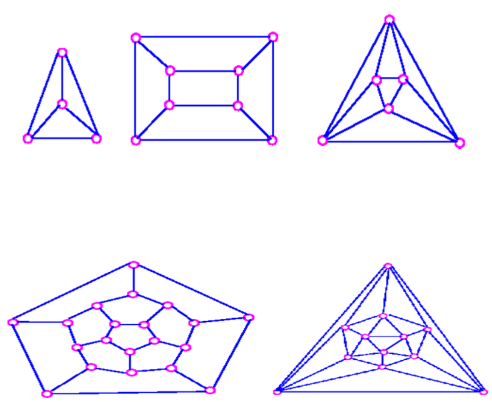
$K_4$



The three plane drawings of $K_4$ are:



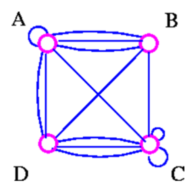
The five Platonic graphs are all planar.



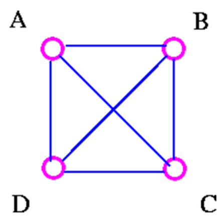
On the other hand, the complete bipartite graph $K_{3,3}$ is not planar, since every drawing of $K_{3,3}$ contains at least one crossing. why? because $K_{3,3}$ has a cycle which must appear in any plane drawing

To study planar graphs, we restrict ourselves to simple graphs.

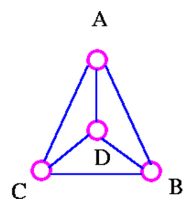
- If a planar graph has multiple edges or loops.
    - Collapse the multiple edges to a single edge.
    - Remove the loops.
- Draw the resulting simple graph without crossing.
- Insert the loops and multiple edges.



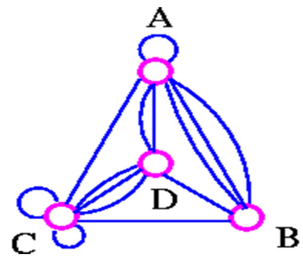

Remove loops and multiple edge.




Draw without multiple edge.



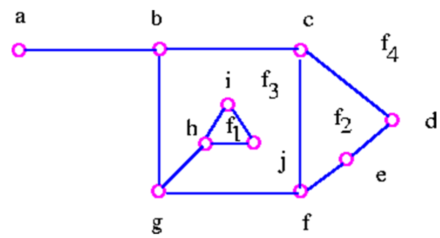

Insert loops and multiple edges.




*B. Euler's Formula*

If G is a planar graph, then any plane drawing of G divides the plane into regions, called faces. One of these faces is unbounded, and is called the infinite face. If $f$ is any face, then the degree of *f*(denoted by deg *f*) is the number of edges encountered in a walk around the boundary of the face *f*. If all faces have the same degree (g, say), the G is face-regular of degree g.

For example, the following graph G has four faces, $f_4$ being the infinite face.




It is easy to see from above graph that deg $f_1$=3, deg $f_2$=4, deg $f_3$=9, deg $f_4$=8.

Note that the sum of all the degrees of the faces is equal to twice the number of edges in the the graph , since each edge either borders two different faces (such as bg, cd, and cf) or occurs twice when walk around a single face (such as ab and gh). The Euler's formula relates the number of vertices, edges and faces of a planar graph. If n, m, and f denote the number of vertices, edges, and faces respectively of a connected planar graph, then we get $n$-$m$+$f$ = 2.

The Euler formula tells us that all plane drawings of a connected planar graph have the same number of faces namely, 2+$m$-$n$.

***Theorem 1 (Euler's Formula)*** *Let G be a connected planar graph, and let* ***n, m*** *and* ***f*** *denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of G. Then* ***n - m + f = 2***.

***Proof*** We employ mathematical induction on edges, m. The induction is obvious for m=0 since in this case n=1 and f=1. Assume that the result is true for all connected plane graphs with fewer than m edges, where m is greater than or equal to 1, and suppose that G has m edges. If G is a tree, then n=m+1 and f=1 so the desired formula follows. On the other hand, if G is not a tree, let e be a cycle edge of G and consider G-e. The connected plane graph G-e has n vertices, m-1 edges, and f-1 faces so that by the inductive hypothesis,

$$\mathbf{n - (m - 1) + (f - 1) = 2}$$

which implies that

$$\mathbf{n - m + f = 2.}$$

We can obtains a number of useful results using Euler's formula. (A "corollary" is a theorem associated with another theorem from which it can be easily derived.)

***Corollary 1*** *Let G be a connected planar simple graph with n vertices, where* $n \geq 3$ *and m edges. Then* $m \leq 3n - 6$.

***Proof*** For graph G with f faces, it follows from the handshaking lemma for planar graph that $2m \geq 3f$ (why?) because the degree of each face of a simple graph is at least 3), so $f \leq 2/3\ m$.

Combining this with Euler's formula

Since $n - m + f = 2$

We get $m - n + 2 \leq 2/3\ m$

Hence $m \leq 3n - 6$.

***As an example of Corollary 1, show that $K_5$ is non-planar.***

***Proof*** Suppose that $K_5$ is a planar graph. Since $K_5$ has 5 vertices and 10 edges it follows from Corollary 1 that 10 $(3 \times 5) - 6 = 9$. This contradiction shows that $K_5$ is non planar.

It is important to note that $K_{3,3}$ has 6 vertices and 9 edges, and it is true that $9 \leq (3 \times 6) - 6 = 12$. This fact simply shows that we cannot use Corollary 1 to prove that $K_{3,3}$ is non-planar. This leads us to following corollary.

***Corollary 2*** *Let G be a connected planar simple graph with n vertices and m edges, and no triangles. Then* $m \leq 2n - 4$.

***Proof*** For graph *G* with *f* faces, it follows from the handshaking lemma for planar graphs that $2m \geq 4f$ (why because the degree of each face of a simple graph without triangles is at least 4), so that $f \leq 1/2\ m$.

Combining this with Euler's formula

Since $n - m + f = 2$

Implies $m - n + 2 = f$

We get $m - n + 2 \leq 1/2m$

Hence $m \leq 2n - 4$

***As an example of Corollary 2, show that $K_{3,3}$ is non-planar.***

***Proof*** Suppose that $K_{3,3}$ is a planar graph. Since $K_{3,3}$ has 6 vertices and 9 edges and no triangles, it follows from Corollary 2 that $9 \leq (2\times6) - 4 = 8$. This contradiction shows that $K_{3,3}$ is non-planar.

***Corollary 3*** Let G be a connected planar simple graph. Then G contains at least one vertex of degree 5 or less.

***Proof*** From Corollary 1, we get $m \leq 3n-6$. Suppose that every vertex in G has degree 6 or more. Then we have $2m \geq 6n$ (why? because 2m is the sum of the vertex-degree), and therefore $m \geq 3n$. This contradiction shows that at least one vertex has degree 5 or less.

Now we will show by using Euler's formula that there are only five regular convex polyhedra - namely, the tetrahedron, cube, octahedron, dodecahedron, and isosahedron.

***Theorem 2*** *There are only 5 regular convex polyhedra.*

***Proof*** We prove this theorem by showing that there are only 5 connected planar graph G with following properties.

i. G is regular of degree d, where $d \geq 3$.
ii. Any plane drawing of G is face-regular of degree g where $g \geq 3$.

Let n, m and f be the numbers of vertices, edges, and faces of such a planar graph G. Then, from properties (i) and (ii), we ge

$$m = 1/2\ dn$$
$$= 1/2\ gf$$

This gives us $n = 2m/d$ and $f = 2m/g$

Here Euler's formula ($n - m + f = 2$) holds, since G is a planar graph.

Therefore, $2m/d - m + 2m/g = 2$

Which can be written as $1/d - 1/2 + 1/g = 1/m$

Since $1/m > 0$, it follows that $1/d + 1/g > 1/2$

Note that each of d and g is at least 3, so each of $1/d$ and $1/g$ is at most $1/3$.

Therefore, $1/d > 1/2 - 1/3 = 1/6$ and

$$1/g > 1/2 - 1/3 = 1/6.$$

and we conclude that $d < 6$ and $g < 6$.

This means that the only possible values of d and g are 3, 4, and 5. However, if both d and g are greater than 3, then

$$1/d + 1/g \leq 1/4 + 1/4 = 1/2$$

which is a contradiction. This leaves us with just five cases:

Case 1: When $d = 3$ and $g = 3$.

we get $1/m = 1/3 - 1/2 + 1/3 = 1/6$

Therefore $m = 6$

It follows that $n = 8$ and $f = 4$ and this gives the Tetrahedron.

Case 2: When $d = 3$ and $g = 4$.

we get $1/m = 1/3 - 1/2 + 1/4 = 1/12$

Therefore $m = 12$

It follows that $n = 8$ and $f = 6$ and this gives the Cube.

Case 3: When $d = 3$ and $g = 5$.

we get $1/m = 1/3 - 1/2 + 1/5 = 1/30$

Therefore $m = 30$

It follows that $n = 20$ and $f = 12$ and this gives the Dodecahedron.

Case 4: When $d = 4$ and $g = 3$.

we get $1/m = 1/4 - 1/2 + 1/3 = 1/12$

Therefore $m = 12$

It follows that $n = 6$ and $f = 8$ and this gives the Octahedron.

Case 5: When d = 5 and g = 3.

we get 1/m = 1/5 - 1/2 + 1/3 = 1/30

Therefore m = 30

It follows that n = 12 and f = 20 and this gives the Icosahedrons.

And this completes the proof.

## II. VECTOR SPACE

Let $V$ be a Euclidean vector space of dimension $2n$ endowed with an inner product (denoted by $\cdot$) and let $F$ be an isometry on $V$ (that is, $Fv\cdot Fw = v\cdot w$, for any $v,w \in V$) such that $F2 = -Id$. Observe that, for any $v \in V$, it is known that $v$ is orthogonal to $Fv$ because $v\cdot Fv = -F2v\cdot Fv = -Fv\cdot v$ and thus $v\cdot Fv = 0$.

In these conditions it is always possible to construct special orthonormal bases of $V$ as follows: let $w1$ be any unit vector of $V$. Then, $Fw1$ is a unit vector too and, moreover, orthogonal to $w1$. Next, if $n > 1$, let $w2$ be any unit vector of $V$ orthogonal to both $w1$ and $Fw1$. It is easy to show that $Fw2$ is another unit vector orthogonal to $w1$, $Fw1$ and $w2$. Continuing this procedure, we get an orthonormal basis $\{w1, \ldots ,w2n\}$ of $V$, where we are denoting $wn+k = Fwk$, $k = 1, \ldots , n$. Further more, we observe that $Fwn+k = -wk$, for any $k = 1, \ldots , n$. The orthonormal bases obtained this way are called *F-bases*.

Conversely, it is also easy to prove that if $\{w1, \ldots ,w2n\}$ is an orthonormal basis of $V$, then there exists a unique isometry $F$ on $V$ such that $F2 = -Id$ and $\{w1, \ldots ,w2n\}$ is an $F$-basis. In fact, $F$ is defined as $Fwk = wn+k$ and $Fwn+k = -wk$, for any $k = 1, \ldots , n$. Now, let $B = \{v1, \ldots , v2n\}$ be an arbitrary orthonormal basis of $V$. We can define a graph $GB$ by following these steps:

1. We consider a vertex for every vector in the basis, labeled with its corresponding natural index. Actually, we will sometimes identify vectors and vertices by using the same notation.
2. We say that the $\{vi, vj\}$ edge exists if and only if $Fvi\cdot vj \_= 0$. Notice that there are no loops in $GB$, since $Fvi\cdot vi = 0$, for any $i = 1, \ldots , 2n$, as we have already pointed out above. We say that a labeled graph $G$ and the basis $B$ are *associated* if $G$ is isomorphic to $GB$.

Now, we are going to present some examples.

***Example 3.1.*** Let $V$ be a 2-dimensional Euclidean vector space and $F$ an isometry of $V$ such that $F2 = -Id$. If we consider an $F$-basis, $\{w1,w2\}$, then it is associated with the graph $K2$ because $Fw1 \cdot w2 = Fw1 \cdot Fw1 = w1 \cdot w1 = 1$.

***Example 3.2*.** Let $V$ be a 4-dimensional Euclidean vector space and $F$ an isometry of $V$ such that $F2 = -Id$ and let us consider an $F$-basis $\{w1,w2,w3,w4\}$. Then, it is associated with the graph $K2 \cup K2$.

Now, let $\theta$ □ $(0, \pi/2)$. If we choose $v1 = \cos\theta w1 + \sin\theta w2$, $v2 = w3$, $v3 = w4$, $v4 = -\sin\theta w1 + \cos\theta w2$, it is easy to see that $\{v1, v2, v3, v4\}$ is an orthonormal basis of $V$ associated with the graph $C4$. Finally, if we choose

$\_v1 = \cos\theta v1 + \sin\theta v2$, $\_v2 = \sin\theta v1 - \cos\theta v2$, $\_v3 = v3$, $\_v4 = v4$, then, $\{\_v1,\_v2,\_v3,\_v4\}$ is an orthonormal basis of $V$ associated with the graph $K4$.

Let be a finite undirected graph G(V,E). The vertex space V(G) of G is the vector space over the finite field of two elements Z/2Z;={0,1} of all functions V→ Z/2Z. Every element of V(G) naturally corresponds the subset of V which assigns a 1 to its vertices. Also every subset of V is uniquely represented in V(G) by its characteristic function. The edge space E(G) is the Z/2Z-vector space freely generated by the edge set E. The dimension of the vertex space is thus the number of vertices of the graph, while the dimension of the edge space is the number of edges. These definitions can be made more explicit. For example, we can describe the edge space as follows:

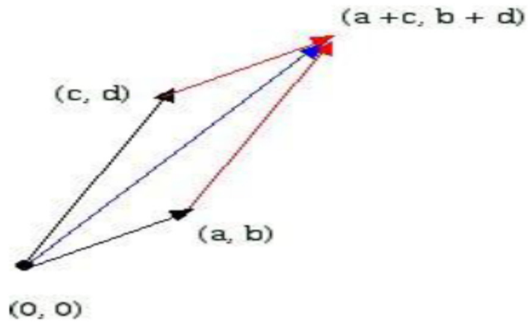
- elements of the vector space are subsets of E, that is, as a set E(G) is the power set of E
- vector addition is defined as the symmetric difference: P+Q=PΔQ , P,Q € E(G)
- scalar multiplication is defined by:

  0.P = 0 P€ E(G)

  1.P = P P€E(G)

The singleton subsets of E form a basis for E(G) and V(G) is the power set of V made into a vector space with similar vector addition and scalar multiplication as defined for E(G).

We can represent velocities, which have a magnitude and direction, by line segments with arrows. In the diagram below, the two original vectors are shown in black. Vectors with the same magnitude and direction as these two are shown drawn in red. The sum of the original two vectors is found by taking the tail of one of the vectors and placing it at the head of the other vector.




## III. Conclusions

A new vector, shown in blue, indicates the sum or resultant of the original two. This is the *parallelogram law* of vector addition which provides the foundation for geometry of vectors.

## IV. ACKNOWLEDGEMENTS

The authors wish to express their gratitude to the referee for his/her very valuable comments and suggestions which have allowed them to improve this paper.

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