FLUCTUATING COUETTE FLOW AND HEAT TRANSFER IN A VERTICAL CHANNEL IN THE PRESENCE OF PERIODIC SUCTION AND UNIFORM INJECTION

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Abstract

Unsteady Couette flow of a viscous incompressible fluid between two vertical parallel plates is considered. Both plates are stationary, one is subjected to a periodic suction and the other is subjected to uniform injection. Due to this type of periodic suction velocity, the flow becomes three-dimensional. The expressions for velocity and temperature distributions are obtained using regular perturbation technique, discussed numerically and shown through graphs. The expressions of skin-friction and Nusselt number at the plates are derived, discussed numerically and their numerical results for various values of physical parameters are shown through graphs.

Key words

Three-dimensionl, Unsteady, Couette flow, injection, suction, skin-friction, Nusselt number.

1.Introduction

Couette flow is important in numerous mechanisms involving the relative motion of two surfaces. The problem of couette flow is considered important in transpiration cooling. Transpiration cooling can very effectively protect certain structural elements in turbojet and rocket engines, like combustion chamber walls, exhaust nozzles or gas turbine blades from hot gases. Eckert and Drake [13] and Jain and Bansal [31] described the reduction of heat transfer in couette flow for the case of an incompressible fluid by injecting the fluid into the flow field from the stationary plate and corresponding removable of heat from the moving plate. The problem is two dimensional due to the uniform injection and suction applied at the porous plates. The solution is well known when both surfaces are flat and moving in their own planes. Gersten and Gross [23] have studied the effect of transverse sinusoidal suction velocity distribution on flow and heat transfer over a plane wall with constant temperature. Gulab Ram and Mishra [15] studied the equations of motion on unsteady MHD flow of conducting fluid through porous medium. Effects of such a suction velocity on various flow and heat transfer problems along horizontal and vertical plates were also been studied extensively by Singh et al. [39], Ahmed and Sharma [30], Chaudhary and Chand [40], Singh and Sharma [20] and Gehlot and Tak [41]. Setayeshpour [5] investigated Magnetohydrodynamic couette flow and heat transfer with variable viscosity and electrical conductivity. An exact solution of Navier-Stokes equations between two parallel plates without suction is well known in Schlichting [16].Unsteady flow through a porous medium bounded by an infinite porous plate subjected to a constant suction and variable temperature was investigated by Raptis [1].Raptis and Perdikis [2] analysed free convective flow through a porous medium bounded by a vertical porous plate with constant suction when the free stream velocity oscillates in time about a constant mean value.

A singular perturbation solution for Couette flow over a semi-infinite porous bed was studied by Hsu and Cheng [10].Chauhan and Shekhawat [12] have analysed heat transfer in Couette flow of a compressible Newtonian fluid in the presence of a naturally permeable boundary. Barlette [6] presented laminar mixed convection with viscous dissipation in a vertical channel. Zaturska et al. [29] have been studied on the flow of a viscous fluid driven along a channel by suction at porous walls. Singh [19] discussed the couette flow between two parallel plates with transverse sinusoidal injection of the fluid at the stationary plate and the constant suction at the plate in motion. Kim [46] concentrated on Unsteady MHD convective heat transfer past semi - infinite vertical porous moving plate with variable suction. Magnetohydrodynamic three-dimensional couette flow with transpiration cooling has been studied by Singh and Sharma [21].Singh and Sharma [22] discussed three-dimensional couette flow through a porous medium with heat transfer. Kamel [27] considered unsteady MHD convection through porous medium with combined heat and mass transfer with heat source/sink. Jothimani and Anjali Devi [42] reported on MHD couette flow with heat transfer and slip flow effects in an inclined channel. Ogulu and Sandile [4] presented radiative heat transfer to MHD couette flow with variable wall temperature. Umavathi and Malashetty [17] observed magnetohydrodynamic mixed convection in a vertical channel. Guria and Jana [26] have studied three dimensional fluctuating couette flow through the porous plates with heat transfer. Three dimensional free convection couette flow with transpiration cooling investigated by Jain and Gupta [32]. Makinde and Osalusi,[33] studied MHD steady flow in a channel with slip at the permeable boundaries. Zanchini [14] investigated mixed convection with variable viscosity in a vertical annulus with uniform wall temperatures. Sharma and Singh [34] analysed unsteady MHD free convective flow and heat transfer along a vertical porous plate with variable suction and internal heat generation. Sharma and Mehta [37] discussed MHD Unsteady slip flow and heat transfer in a channel with slip at the permeable boundaries. Vidhya and Sundarammal Kesavan [28] considered laminar convection through porous medium between two vertical parallel plates with heat source. Khem Chand [25] presented heat transfer in three dimensional MHD boundary layer flow over a continuous porous surface moving in a parallel free stream. Das et al. [45] described effect of heat source on MHD free convection flow past an oscillating porous plate in the slip flow regime. Gireesha et al. [7] considered three-dimensional Couette flow of a dusty fluid with heat transfer. Jha and Apere [8] observed time-dependent MHD Couette flow of rotating fluid with Hall and ion-slip currents. Solution of MHD oscillatory convection flow through porous medium in a vertical porous channel in slip flow regime was investigated by Singh, [18].Khem Chand and Sapna [24] explained Hydromagnetic free convective oscillatory couette flow through a porous vertical channel with periodic wall temperature .Sharma and Dadheech described [35] Effect of volumetric heat generation / absorption on convective heat and mass transfer in porous medium in between two vertical plates . Das et al. [44] investigated unsteady hydromagnetic convective flow past an infinite vertical porous plate in a porous medium. Kesavaiah et al. [11] studied effects of radiation and free convection currents on unsteady couette flow between two vertical parallel plates with constant heat flux and heat source through porous medium. Das et al. [43] presented effect of variable suction and radiative heat transfer on MHD couette flow through a porous medium in the slip flow regime.Mishra [3] observed transient free convection flow past a vertical plate through porous medium with variation in slip flow regime. Sharma and Sharma [36] presented oscillatory MHD Free Convective Flow and Mass transfer through Porous Medium bounded by a Vertical Porous Channel with Thermal Source and Slip Boundary Conditions. Sharma and Yadav [38] discussed analysis of MHD convective flow along a moving semivertical plate with internal heat generation. Swarnalathamma and Veera Krishna [9] described peristaltic hemodynamic flow of couple stress fluid through a porous medium under the influence of magnetic field with slip effect.Zhang [47] explained effect of wall surface modification in the combined Couette and Poiseuille flows in a nano channel.

The aim of present paper is to investigate the unsteady couette flow and heat transfer between two vertical parallel porous flat plates with periodic suction at one plate and constant injection at another plate, keeping in view that both plates are stationary. We assume that the periodic suction velocity is time-dependent and perpendicular to the flow direction. Due to the periodic suction, the flow becomes three-dimensional. The main flow velocity profile and shear stress have been calculated and plotted. The heat transfer characteristic has also been studied on taking uniform volumetric rate of heat generation/absorption into account.

2.Formulation of the Problem

Consider the unsteady flow of a viscous incompressible fluid between two vertical flat plates separared by a distance d, keeping in view that both plates are stationary. One plate is subjected to a constant injection $-V_0$ and the other plate to a transverse sinusoidal suction velocity distribution of the form

$$v^* = -V_0\left[1+\varepsilon\cos\left(\frac{\pi z^*}{d}-ct^*\right)\right], \qquad \ldots(1)$$

where $\varepsilon(<<1)$ is the amplitude of the suction velocity. The plates are assumed to be at constant temperature T_0 and T_1, respectively. All the physical quantities are independent of x^* for this problem of fully developed laminar flow but the flow remains three dimensional due to the periodic suction velocity as given in eqn.(1). Denoting velocity components u^*, v^*, w^* in the direction x^*-, y^*-, z^*-axes, respectively, the flow is governed the following equations:

$$\frac{\partial v^*}{\partial y^*}+\frac{\partial w^*}{\partial z^*}=0, \qquad \text{....(2)}$$

$$\frac{\partial u^*}{\partial t^*}+v^*\frac{\partial u^*}{\partial y^*}+w^*\frac{\partial u^*}{\partial z^*}=g\beta(T^*-T_0)+\nu\left(\frac{\partial^2 u^*}{\partial y^{*2}}+\frac{\partial^2 u^*}{\partial z^{*2}}\right), \qquad \text{....(3)}$$

$$\frac{\partial v^*}{\partial t^*}+v^*\frac{\partial v^*}{\partial y^*}+w^*\frac{\partial v^*}{\partial z^*}=-\frac{1}{\rho}\frac{\partial p^*}{\partial y^*}+\nu\left(\frac{\partial^2 v^*}{\partial y^{*2}}+\frac{\partial^2 v^*}{\partial z^{*2}}\right), \qquad \text{....(4)}$$

$$\frac{\partial w^*}{\partial t^*}+v^*\frac{\partial w^*}{\partial y^*}+w^*\frac{\partial w^*}{\partial z^*}=-\frac{1}{\rho}\frac{\partial p^*}{\partial z^*}+\nu\left(\frac{\partial^2 w^*}{\partial y^{*2}}+\frac{\partial^2 w^*}{\partial z^{*2}}\right), \qquad \text{....(5)}$$

$$\frac{\partial T^*}{\partial t^*}+v^*\frac{\partial T^*}{\partial y^*}+w^*\frac{\partial T^*}{\partial z^*}=\frac{\kappa}{\rho C_p}\left(\frac{\partial^2 T^*}{\partial y^{*2}}+\frac{\partial^2 T^*}{\partial z^{*2}}\right)+\frac{Q^*}{\rho C_p}(T^*-T_0), \qquad \text{....(6)}$$

where ν is the kinematic viscosity, ρ the density, p^* the fluid pressure and Q* the volumetric rate of heat generation/absorption.

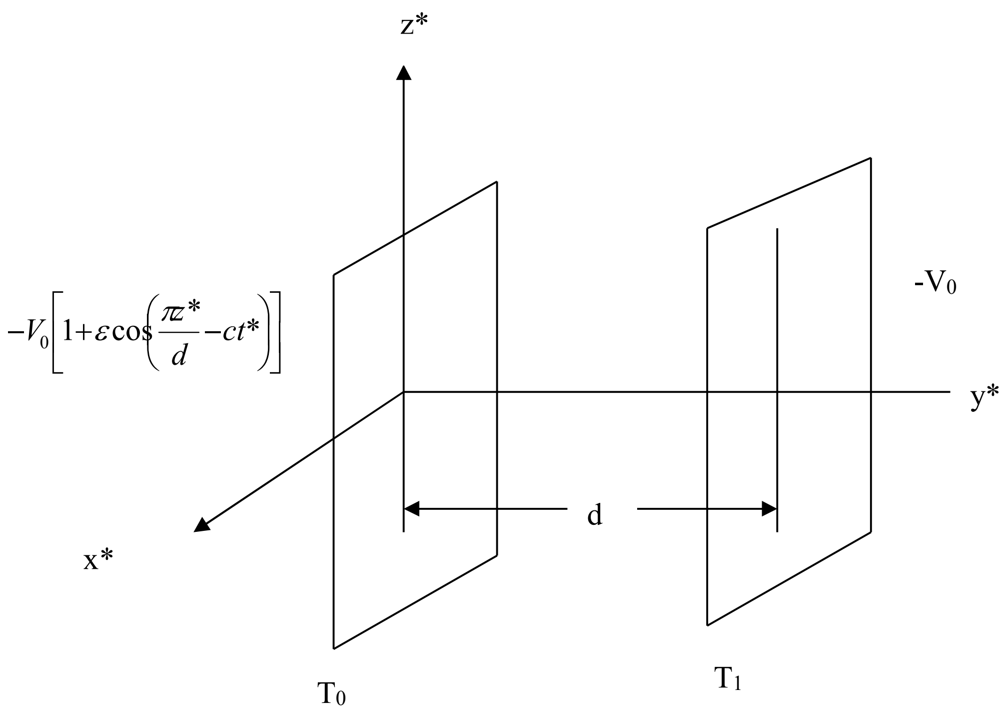

Figure 1. Physical Model

The boundary conditions of the problem are

$$y^* = 0 : u^* = 0, v^* = -V_0\left(1 + \cos\left\{\frac{\pi}{d} z^* - ct^*\right\}\right), w^* = 0, T^* = T_0,$$

$$y^* = h : u^* = 0, v^* = -V_0, w^* = 0, T^* = T_1 (T_1 > T_0). \quad \text{….(7)}$$

3.Method of Solution

Introducing the following non-dimensional quantities

$$y = \frac{y^*}{d},\ z = \frac{z^*}{d},\ t = ct^*,\ p = \frac{p^*}{\rho V_0^2},\ u = \frac{u^*}{V_0},\ v = \frac{v^*}{V_0},\ w = \frac{w^*}{V_0},\ \theta = \frac{T^* - T_0}{T_1 - T_0} \quad \text{….(8)}$$

into the equations (2) to (6), we get

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad \text{….(9)}$$

$$\omega\frac{\partial u}{\partial t} + \mathrm{Re}\left(v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \mathrm{Re}\, Gr\,\theta + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}, \quad \text{….(10)}$$

$$\omega\frac{\partial v}{\partial t} + \mathrm{Re}\left(v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\mathrm{Re}\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}, \quad \text{….(11)}$$

$$\omega\frac{\partial w}{\partial t} + \mathrm{Re}\left(v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\mathrm{Re}\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}, \quad \text{….(12)}$$

$$\mathrm{Pr}\,\omega\frac{\partial \theta}{\partial t} + \mathrm{Re}\,\mathrm{Pr}\left(v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) = \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} + Q\theta, \quad \text{….(13)}$$

where $\mathrm{Re} = \frac{V_0 d}{\nu}$, the Reynolds number ; $Gr = \frac{g\beta d(T_1 - T_0)}{V_0^2}$, the Grashoff number ;

$\mathrm{Pr} = \frac{\mu C_p}{\kappa}$, the Prandtl number ; $\omega = \frac{cd^2}{\nu}$, the frequency parameter and $Q = \frac{Q^* d^2}{\kappa}$ the rate of heat generation/absorption parameter.

The boundary conditions (7) in non-dimensional form are

$$y = 0 : u = 0, v = -(1 + \varepsilon\cos\{\pi z - t\}), w = 0, \theta = 0,$$

$$y = 1 : u = 0, v = -1, w = 0, \theta = 1. \quad \text{….(14)}$$

Since ε (<<1) is very small, Therefore assuming

$$f(y,z) = f_0(y) + \varepsilon f_1(y,z,t) + \varepsilon^2 f_2(y,z,t) + \ldots\ldots. \quad \text{….(15)}$$

where f stands for u, v, w, p or θ .

when ε=0 , the problem reduces to the two-dimensional flow with constant suction and injection at both the plates. In this case eqns. (9) to (13) reduces to

$$\frac{\partial v_0}{\partial y} = 0, \qquad \ldots(16)$$

$$\frac{\partial^2 w_0}{\partial y^2} - \mathrm{Re}\, v_0 \frac{\partial w_0}{\partial y} = 0, \qquad \ldots(17)$$

$$\frac{\partial^2 \theta_0}{\partial y^2} - \mathrm{Re}\,\mathrm{Pr}\, v_0 \frac{\partial \theta_0}{\partial y} + Q\theta_0 = 0, \qquad \ldots(18)$$

$$\frac{\partial^2 u_0}{\partial y^2} - \mathrm{Re}\, v_0 \frac{\partial u_0}{\partial y} = -\mathrm{Re}\, Gr\theta_0, \qquad \ldots(19)$$

and the corresponding boundary conditions are

$$y = 0 : u_0 = 0, v_0 = -1, \theta_0 = 0,$$

$$y = 1 : u_0 = 0, v_0 = -1, \theta_0 = 1. \qquad \ldots(20)$$

The solution of eqns. (16) to (18) using boundary conditions (19) is given by

$$v_0(y) = -1,\ w_0 = 0,\ \theta_0 = C_1 e^{m_1 y} + C_2 e^{m_2 y}, \text{and } u_0 = C_3 + C_4 e^{-\mathrm{Re}\, y} + A_2 e^{m_1 y} + A_3 e^{m_2 y}. \qquad \ldots(21)$$

When $\varepsilon \neq 0$, for small values of ε, we take only upto $O(\varepsilon)$. Substituting (15) in eqns. (9) to (13) and comparing the coefficients of ε from both sides, we get

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \qquad \ldots(22)$$

$$\omega \frac{\partial u_1}{\partial t} + \mathrm{Re}\left(v_0 \frac{\partial u_1}{\partial y} + w_0 \frac{\partial u_1}{\partial z} + v_1 \frac{\partial u_0}{\partial y} \right) = \mathrm{Re}\, Gr\theta_1 + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}, \qquad \ldots(23)$$

$$\omega \frac{\partial v_1}{\partial t} + \mathrm{Re}\left(v_0 \frac{\partial v_1}{\partial y} + w_0 \frac{\partial v_1}{\partial z} + v_1 \frac{\partial v_0}{\partial y} \right) = -\mathrm{Re}\frac{\partial p_1}{\partial y} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}, \qquad \ldots(24)$$

$$\omega \frac{\partial w_1}{\partial t} + \mathrm{Re}\left(v_0 \frac{\partial w_1}{\partial y} + w_0 \frac{\partial w_1}{\partial z} + v_1 \frac{\partial w_0}{\partial y} \right) = -\mathrm{Re}\frac{\partial p_1}{\partial z} + \frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}, \qquad \ldots(25)$$

$$\mathrm{Pr}\,\omega \frac{\partial \theta_1}{\partial t} + \mathrm{Re}\,\mathrm{Pr}\left(v_1 \frac{\partial \theta_0}{\partial y} + w_0 \frac{\partial \theta_1}{\partial z} + v_0 \frac{\partial \theta_1}{\partial y} \right) = \frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2} + Q\theta_1. \qquad \ldots(26)$$

The corresponding boundary conditions are reduced to

$$y = 0 : u_1 = 0, v_1 = -\cos(\pi z - t), w_1 = 0, \theta_1 = 0,$$

$$y = 1 : u_1 = 0, v_1 = 0, w_1 = 0, \theta_1 = 0. \qquad \ldots(27)$$

These are the linear partial differential equations describing the three-dimensional flow. To solve eqns. (23) to (26) we assume u_1, v_1, w_1, p_1 and θ_1 of the following form

$$u_1(y,z,t) = u_{11}(y)\{\cos(\pi z - t) + i\sin(\pi z - t)\},$$

$$v_1(y,z,t) = v_{11}(y)\{\cos(\pi z - t) + i\sin(\pi z - t)\},$$

$$w_1(y,z,t) = \frac{i}{\pi} v'_{11}(y)\{\cos(\pi z - t) + i\sin(\pi z - t)\},$$

$$p_1(y,z,t) = p_{11}(y)\{\cos(\pi z - t) + i\sin(\pi z - t)\},$$

$$\theta_1(y,z,t) = \theta_{11}(y)\{\cos(\pi z - t) + i\sin(\pi z - t)\}. \quad \text{....(28)}$$

The corresponding boundary conditions are

$$y = 0 : u_{11} = 0, v_{11} = -1, w_{11} = 0, \theta_{11} = 0,$$

$$y = 1 : u_{11} = 0, v_{11} = 0, w_{11} = 0, \theta_{11} = 0. \quad \text{....(29)}$$

Substituting eqn. (28) in eqns. (23) to (26) we get

$$u''_{11} + \mathrm{Re}\, u'_{11} - (\pi^2 - i\omega)u_{11} = \mathrm{Re}\, v_{11}u'_0 - \mathrm{Re}\, Gr\,\theta_{11}, \quad \text{....(30)}$$

$$v''_{11} + \mathrm{Re}\, v'_{11} - (\pi^2 - i\omega)v_{11} = \mathrm{Re}\, p'_{11}, \quad \text{....(31)}$$

$$v'''_{11} + \mathrm{Re}\, v''_{11} - (\pi^2 - i\omega)v'_{11} = \mathrm{Re}\,\pi^2 p_{11}, \quad \text{....(32)}$$

$$\theta''_{11} + \mathrm{Re}\Pr\theta'_{11} - (\pi^2 - Q - i\omega\Pr)\theta_{11} = \mathrm{Re}\Pr v_{11}\theta'_0. \quad \text{....(33)}$$

Solving eqns. (30) to (33) under the conditions (29), we get

$$\begin{aligned} u_{11}(y) = \{ & C_9 e^{(K_3+iK_4)y} + C_{10} e^{(K_5+iK_6)y} + (D_{47}+iD_{48})e^{(D_1-iG_4)y} + (D_{49}+iD_{50})e^{(D_2-iG_6)y} \\ & (D_{55}+iD_{56})e^{(D_5-iG_4)y} + (D_{57}+iD_{58})e^{(D_6-iG_6)y} + (D_{63}+iD_{64})e^{(D_9-iG_4)y} \\ & + (D_{65}+iD_{66})e^{(D_{10}-iG_6)y} + (D_{71}+iD_{72})e^{(H_3+iH_4)y} + (D_{73}+iD_{74})e^{(H_5+iH_6)y} \\ & + (D_{75}+iD_{76})e^{(H_7+iH_8)y} + (D_{77}+iD_{78})e^{(H_9+iH_{10})y} + (D_{83}+iD_{84})e^{(J_7+iJ_8)y} \\ & + (D_{85}+iD_{86})e^{(J_9+iJ_{10})y} + (D_{51}+iD_{52})e^{D_3 y} + (D_{53}+iD_{54})e^{D_4 y} + (D_{91}+iD_{92})e^{D_7 y} \\ & + (D_{93}+iD_{94})e^{D_8 y} + (D_{95}+iD_{96})e^{D_{11} y} + (D_{97}+iD_{98})e^{D_{12} y}\}, \end{aligned}$$

$$\text{....(34)}$$

$$v_{11}(y) = [Ae^{-r_1 y} + Be^{-r_2 y} + Ce^{\pi y} + De^{-\pi y}], \quad \text{....(35)}$$

$$w_{11}(y) = -\frac{i}{\pi}[Ar_1 e^{-r_1 y} + Br_2 e^{-r_2 y} - C\pi e^{\pi y} + D\pi e^{-\pi y}], \quad \text{....(36)}$$

$$\begin{aligned} \theta_{11}(y) = \{ & C_7 e^{(H_3+iH_4)y} + C_8 e^{(H_5+iH_6)y} + \mathrm{Re}\Pr C_1 m_1[(H_{17}+iH_{18})e^{(H_7+iH_8)y} + (H_{21}+iH_{22})e^{(H_9+iH_{10})y} \\ & + (H_{25}+iH_{26})e^{(m_1+\pi)y} + (H_{29}+iH_{30})e^{(m_1-\pi)y}] + \mathrm{Re}\Pr C_2 m_2[(J_{17}+iJ_{18})e^{(J_7+iJ_8)y} + \\ & (J_{21}+iJ_{22})e^{(J_9+iJ_{10})y} + (J_{25}+iJ_{26})e^{(m_2+\pi)y} + (J_{29}+iJ_{30})e^{(m_2-\pi)y}]\}. \end{aligned} \quad \text{....(37)}$$

On separating real and imaginary parts, we get

$$u_{11R} = e^{K_3 y}(D_{109}\cos K_4 y - D_{110}\sin K_4 y) + e^{K_5 y}(D_{107}\cos K_6 y - D_{108}\sin K_6 y) + D_{101}(y),$$

$$u_{11I} = e^{K_3 y}(D_{110}\cos K_4 y + D_{109}\sin K_4 y) + e^{K_5 y}(D_{108}\cos K_6 y + D_{107}\sin K_6 y) + D_{102}(y),$$

$$u_1 = u_{11R}\cos(\pi z - t) - u_{11I}\sin(\pi z - t), \qquad \ldots(38)$$

$$v_{11R} = e^{-G_3 y}(G_{19}\cos G_4 y + G_{20}\sin G_4 y) + e^{-G_5 y}(G_{23}\cos G_6 y + G_{24}\sin G_6 y) + G_{25}e^{\pi y} + G_{27}e^{-\pi y},$$

$$v_{11I} = e^{-G_3 y}(G_{20}\cos G_4 y - G_{19}\sin G_4 y) + e^{-G_5 y}(G_{24}\cos G_6 y - G_{23}\sin G_6 y) + G_{26}e^{\pi y} + G_{28}e^{-\pi y},$$

$$v_1 = v_{11R}\cos(\pi z - t) - v_{11I}\sin(\pi z - t), \qquad \ldots(39)$$

$$\theta_{11R} = e^{H_3 y}(H_{41}\cos H_4 y - H_{42}\sin H_4 y) + e^{H_5 y}(H_{39}\cos H_6 y - H_{40}\sin H_6 y) + H_{33}(y),$$

$$\theta_{11I} = e^{H_3 y}(H_{42}\cos H_4 y + H_{41}\sin H_4 y) + e^{H_5 y}(H_{40}\cos H_6 y + H_{39}\sin H_6 y) + H_{34}(y),$$

$$\theta_1 = \theta_{11R}\cos(\pi z - t) - \theta_{11I}\sin(\pi z - t), \qquad \ldots(40)$$

Finally , the expressions of $u(y,z,t)$ and $\theta(y,z,t)$ are known in the following form

$$u(y,z,t) = u_\circ(y) + \varepsilon\{u_{11R}(y)\cos(\pi z - t) - u_{11I}(y)\sin(\pi z - t)\}, \qquad \ldots(41)$$

$$\theta(y,z,t) = \theta_\circ(y) + \varepsilon\{\theta_{11R}(y)\cos(\pi z - t) - \theta_{11I}(y)\sin(\pi z - t)\}, \qquad \ldots(42)$$

where

$u_{11R}(y)$ = Real part $\{u_{11R}(y)\}$, $u_{11I}(y)$ = Imaginary part $\{u_{11I}(y)\}$,

$\theta_{11R}(y)$ = Real part $\{\theta_{11R}(y)\}$, $\theta_{11I}(y)$ = Imaginary part $\{\theta_{11I}(y)\}$,

$$H_{33}(y) = \mathrm{Re}\,\mathrm{Pr}\,C_1 m_1\{e^{H_7 y}(H_{17}\cos H_8 y - H_{18}\sin H_8 y) + e^{H_9 y}(H_{21}\cos H_{10} y - H_{22}\sin H_{10} y)$$
$$+ \mathrm{Re}\,\mathrm{Pr}\,C_2 m_2\{e^{J_7 y}(J_{17}\cos J_8 y - J_{18}\sin J_8 y) + e^{J_9 y}(J_{21}\cos J_{10} y - J_{22}\sin J_{10} y)$$
$$+ e^{(m_1+\pi)y}H_{25} + e^{(m_1-\pi)y}H_{29} + e^{(m_2+\pi)y}J_{25} + e^{(m_2-\pi)y}J_{29},$$

$$H_{34}(y) = \mathrm{Re}\,\mathrm{Pr}\,C_1 m_1\{e^{H_7 y}(H_{18}\cos H_8 y + H_{17}\sin H_8 y) + e^{H_9 y}(H_{22}\cos H_{10} y + H_{21}\sin H_{10} y)$$
$$+ \mathrm{Re}\,\mathrm{Pr}\,C_2 m_2\{e^{J_7 y}(J_{18}\cos J_8 y + J_{17}\sin J_8 y) + e^{J_9 y}(J_{22}\cos J_{10} y + J_{21}\sin J_{10} y)$$
$$+ e^{(m_1+\pi)y}H_{26} + e^{(m_1-\pi)y}H_{30} + e^{(m_2+\pi)y}J_{26} + e^{(m_2-\pi)y}J_{30},$$

$$D_{101}(y) = e^{D_1 y}(D_{47}\cos G_4 y + D_{48}\sin G_4 y) + e^{D_2 y}(D_{49}\cos G_6 y + D_{50}\sin G_6 y))$$
$$+ e^{D_5 y}(D_{55}\cos G_4 y + D_{56}\sin G_4 y + e^{D_6 y}(D_{57}\cos G_6 y + D_{58}\sin G_6 y) + e^{D_9 y}(D_{63}\cos G_4 y + D_{64}\sin G_4 y)$$
$$+ e^{D_{10} y}(D_{65}\cos G_6 y + D_{66}\sin G_6 y) + e^{H_3 y}(D_{71}\cos H_4 y - D_{72}\sin H_4 y) + e^{H_5 y}(D_{73}\cos H_6 y - D_{74}\sin H_6 y)$$
$$+ e^{H_7 y}(D_{75}\cos H_8 y - D_{76}\sin H_8 y) + e^{H_9 y}(D_{77}\cos H_{10} y - D_{78}\sin H_{10} y) + e^{J_7 y}(D_{83}\cos J_8 y - D_{84}\sin J_8 y)$$
$$+ e^{J_9 y}(D_{85}\cos J_{10} y - D_{86}\sin J_{10} y) + e^{D_3 y}D_{51} + e^{D_4 y}D_{53} + e^{D_7 y}D_{91} + e^{D_8 y}D_{93} + e^{D_{11} y}D_{95} + e^{D_{12} y}D_{97},$$

$$D_{102}(y) = e^{D_1 y}(D_{48}\cos G_4 y - D_{47}\sin G_4 y) + e^{D_2 y}(D_{50}\cos G_6 y - D_{49}\sin G_6 y)$$
$$+ e^{D_5 y}(D_{56}\cos G_4 y - D_{55}\sin G_4 y) + e^{D_6 y}(D_{58}\cos G_6 y - D_{57}\sin G_6 y) + e^{D_9 y}(D_{64}\cos G_4 y - D_{63}\sin G_4 y)$$
$$+ e^{D_{10} y}(D_{66}\cos G_6 y - D_{65}\sin G_6 y) + e^{H_3 y}(D_{72}\cos H_4 y + D_{71}\sin H_4 y) + e^{H_5 y}(D_{74}\cos H_6 y + D_{73}\sin H_6 y)$$
$$+ e^{H_7 y}(D_{76}\cos H_8 y + D_{75}\sin H_8 y) + e^{H_9 y}(D_{78}\cos H_{10} y + D_{77}\sin H_{10} y) + e^{J_7 y}(D_{84}\cos J_8 y + D_{83}\sin J_8 y)$$
$$+ e^{J_9 y}(D_{86}\cos J_{10} y + D_{85}\sin J_{10} y) + e^{D_3 y}D_{52} + e^{D_4 y}D_{54} + e^{D_7 y}D_{92} + e^{D_8 y}D_{94} + e^{D_{11} y}D_{96} + e^{D_{12} y}D_{98},$$

4. Skin-friction coefficient

The non-dimensional shearing stress on the surface of a body due to a fluid motion is known as Skin-friction coefficient.

Skin-friction coefficient due to the main flow at both the plates is given by

$$C_f = \frac{\tau_\omega d}{\mu V_0} = \left(\frac{\partial u}{\partial y}\right)_{y=0,1}, \quad \ldots(43)$$

where $\tau_\omega = \mu\left(\frac{\partial u^*}{\partial y^*}\right)_{y^*=0,h}$.

shear stress τ_ω is proportional to shear rate with a viscosity constant or viscosity function.

5. Nusselt number

The non-dimensioanl rate of heat transfer quantity is termed as Nusselt number. The rate of heat transfer in terms of Nusselt number at both the plates is given by

$$Nu = \frac{qd}{\kappa(T_1 - T_0)} = -\left(\frac{\partial \theta}{\partial y}\right)_{y=0,1}, \quad \ldots(44)$$

where $q = -\kappa\left(\frac{\partial T^*}{\partial y^*}\right)_{y^*=0,h}$.

It is based on Fourier’s law of heat conduction and heat is transferred from higher temperature to lower temperature , that’s why negative sign is introduced in the equation of Fourier’s law of heat conduction.

6. Results and Discussion

Present paper described the unsteady three dimensional flow of a viscous incompressible fluid between two vertical flat plates separated by a fixed distance d, but both plates are kept stationary. One plate is subjected to a constant injection and the other plate to a transverse sinusoidal suction velocity distribution. The plates are assumed to be at constant temperature T_0 and T_1, respectively. The velocity and temperature profiles have been plotted in (Fig.2-7) to study the effect of different non-dimensional parameters such as effect of Grashof number , Reynolds number , Prandtl number, frequency parameter ,heat source parameter or with the lapse of time are described. Furthermore, skin friction and Nusselt number have been calculated for different values of non-dimensional parameters.

It is observed from figure 2 that fluid velocity increases due to increase in cross-flow Reynolds number, the Grashof number, heat source parameter, frequency or with the lapse of time, while it decreases due to increase in the Prandtl number or in case of sink.

Fluid temperature decreases due to increase in cross-flow Reynolds number, the Prandtl number, heat source parameter or in case of sink, while it increases due to increase in the frequency or with the lapse of time as seen from fgure 3.

Skin-friction coefficient at the plate (when y = 0) increases due to increase in cross-flow Reynolds number, the Grashof number, heat source parameter, frequency or with the lapse of time, while it decreases due to increase in the Prandtl number or in case of sink as noted from figure 4.

Figure 5 depicts that skin-friction coefficient at the plate (when y = 1) decreases due to increase in cross-flow Reynolds number, the Grashof number, heat source parameter, frequency or with the lapse of time, while it increases due to increase in the Prandtl number.

It is noted from figure 6 that the Nusselt number at the plate (when y = 0) decreases due to increase in the Prandtl number, frequency, with the lapse of time or cross-flow Reynolds number, while it increases with the increase in the cross-flow Reynolds number for large value of heat source parameter i.e. Q = 10.

It is seen from figure 7 that the Nusselt number at the plate (when y = 1) decreases due to increase in the Prandtl number, frequency, cross-flow Reynolds number, while it increases due to increase in the heat source parameter, with the lapse of time or due to increase in cross-flow Reynolds number in the presence of sink. The Nusselt number at the plate (when y = 1) has positive values except in case of sink.

The increase of Reynolds number means the increase of inertial forces because of which velocity increases.The increase of velocity with the increase of Grashof number physically means that the enhancement of the buoyancy force leads to increase the velocity component. Prandtl number is the ratio of kinematic viscosity and thermal diffusivity . Increase in prandtl number implies increase in viscosity of the fluid which makes the fluid thick and hence decrease in velocity.We can also observe that if Prandtl number increases the thermal diffusivity decreases and this phenomenon lead to the decreasing manner of the energy transfer ability that reduces the thermal boundary layer. Skin friction coefficient at the plate (when y = 0) shows exactly opposite behavior at the plate (when y = 1), for various values of physical parameters. Nusselt number at the plate (when y = 0) shows exactly same behavior at the plate (when y = 1), for various values of physical parameters like Prandtl number, frequency and heat source.

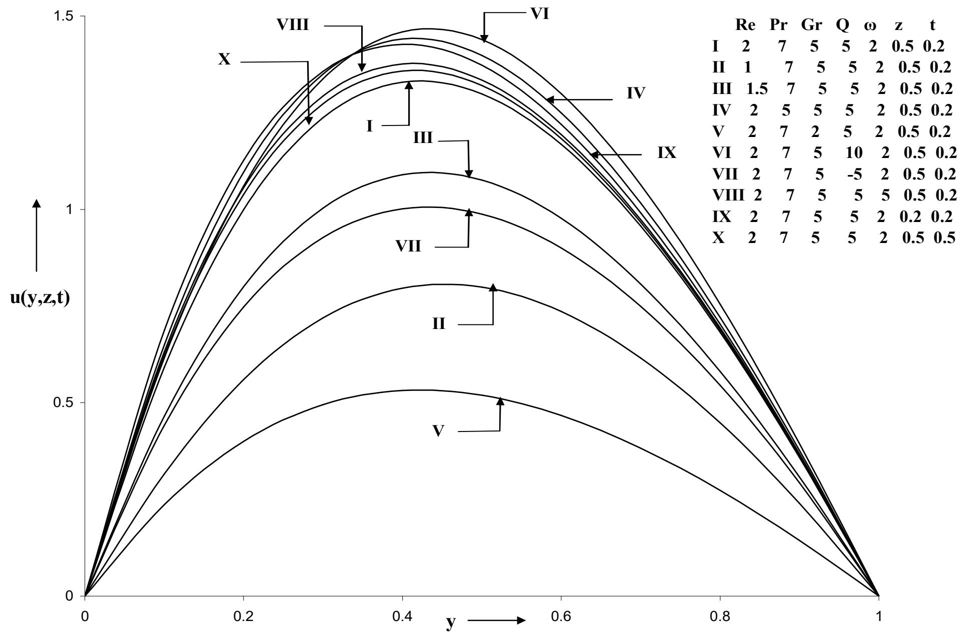

Figure 2. Main velocity distribution u versus y when ε=0.2.

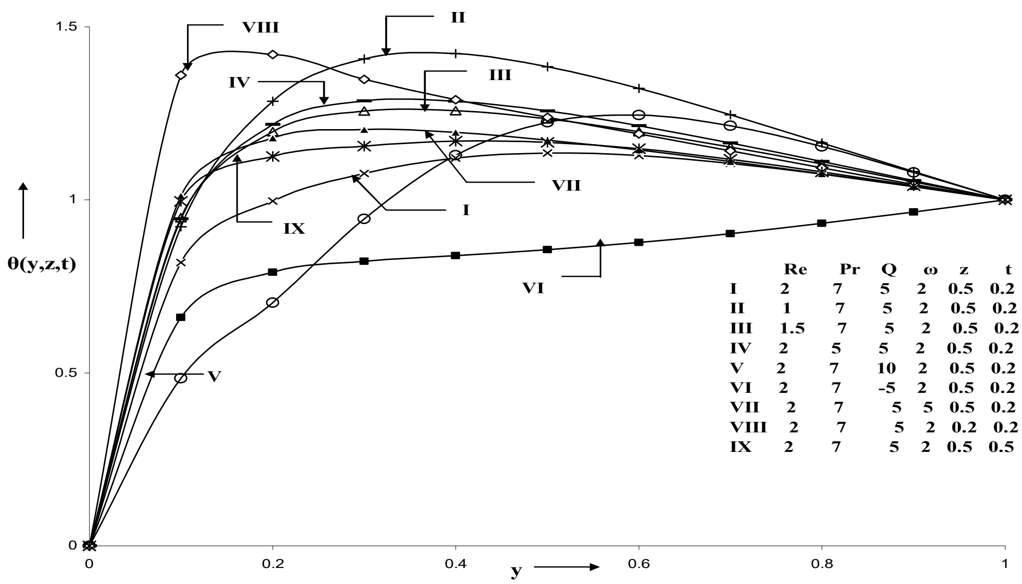

Figure 3. Temperature distribution versus y when ε=0.2.

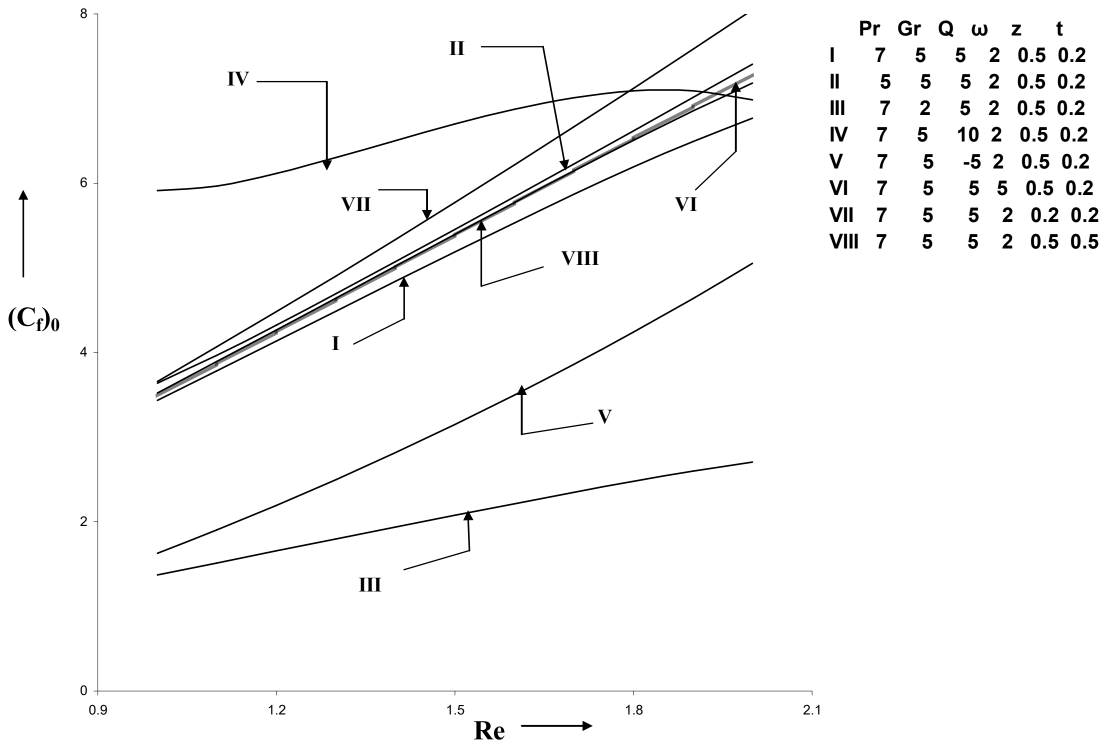

Figure 4.Skin-friction coefficient $(C_f)_0$ at the plate $y = 0$ versus Re when ε=0.2.

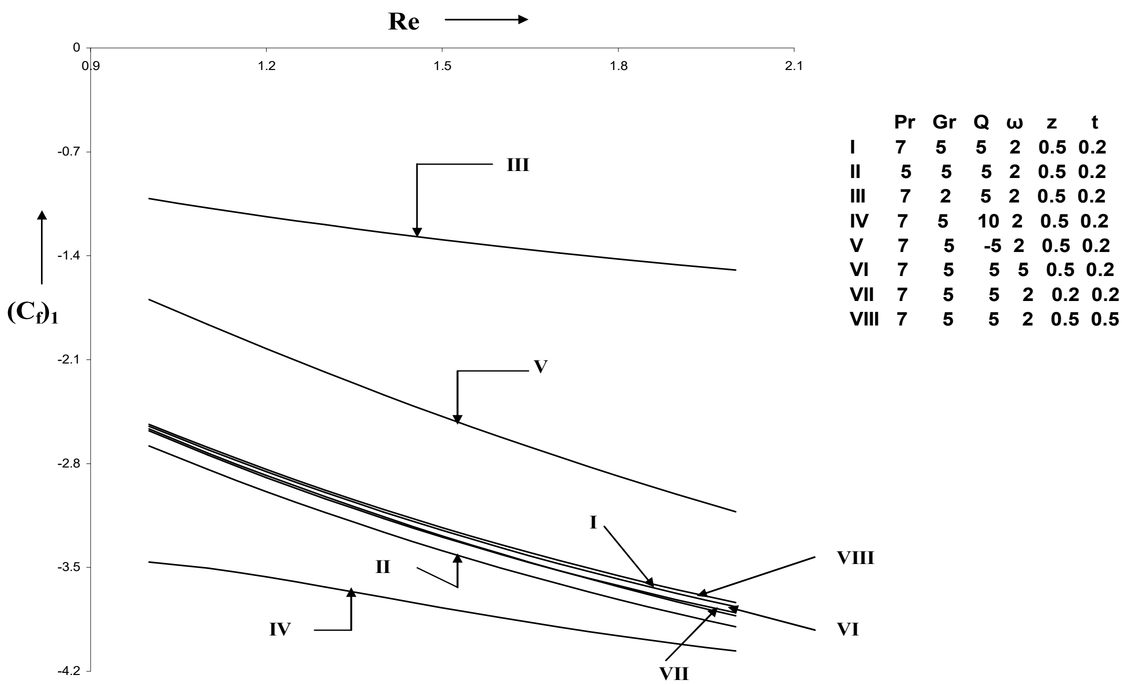

Figure 5.Skin-friction coefficient $(C_f)_1$ at the plate $y = 1$ versus Re when ε=0.2.

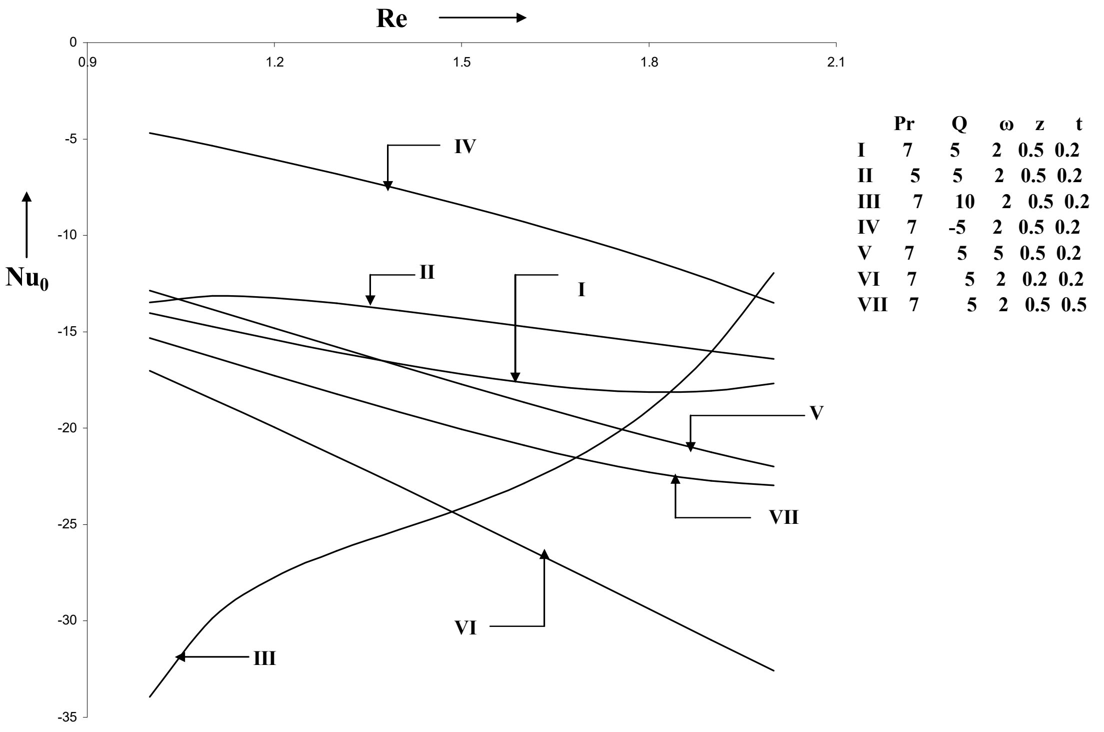

Figure 6. Nusselt number Nu_0 at the plate y = 0 versus Re when ε=0.2.

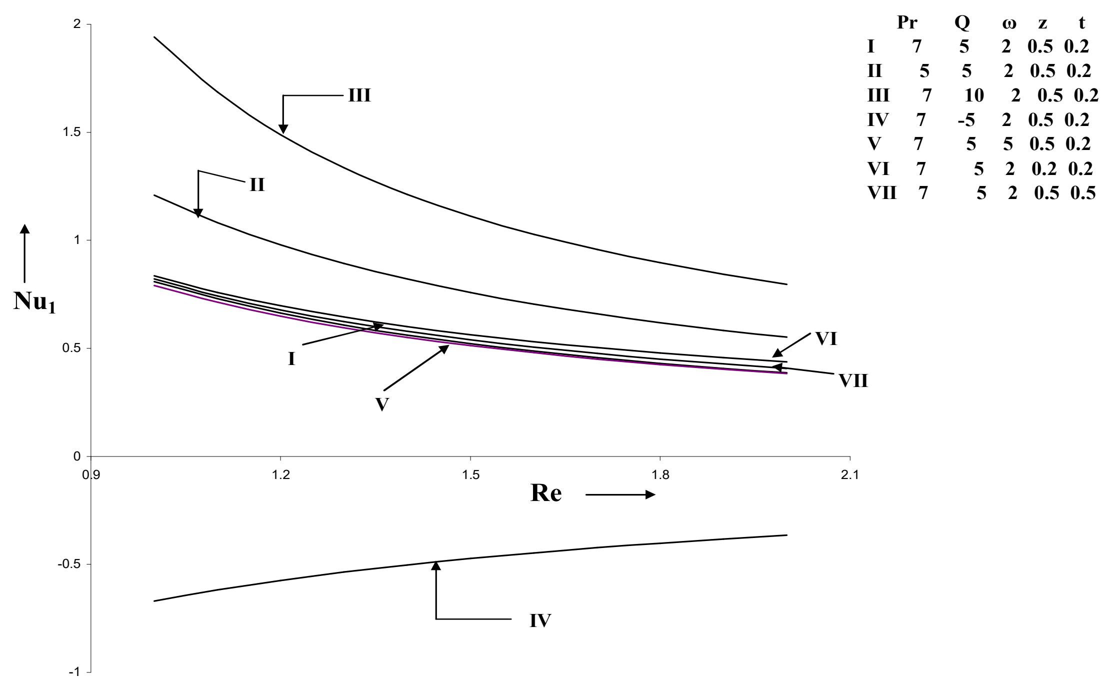

Figure 7. Nusselt number Nu_1 at the plate y = 1 versus Re when ε=0.2.

7. Conclusions:

(i) Fluid velocity increases due to increase of buoyancy force, heat source,frequency or cross-flow velocity, while it decreases due to increase in the Prandtl number.

(ii) Fluid temperature increases due to increase in frequency, while it decreases due to increase in the Prandtl number, cross-flow velocity, or heat source\sink parameter.

(iii) Skin-friction coefficient at the plate (when y = 0) increases due to increase in heat source, frequency or buoyancy force, while it decreases due to increase in the Prandtl number.

(iv) Skin-friction coefficient at the plate (when y = 1) increases due to increase in the Prandtl number, while it decreases due to increase in heat source, frequency or buoyancy force .

(v) Nusselt number at the plate (when y = 0) increases due to increase in heat source, while it decreases due to increase in frequency or the Prandtl number.

(vi) Nusselt number at the plate (when y = 1) increases due to increase in heat source, while it decreases due to increase in the Prandtl number or frequency.

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Appendix

$$m_1 = \frac{-\operatorname{Re}\Pr + \sqrt{\operatorname{Re}^2 \Pr^2 - 4Q}}{2},\ m_2 = \frac{-\operatorname{Re}\Pr - \sqrt{\operatorname{Re}^2 \Pr^2 - 4Q}}{2},\ C_2 = \frac{1}{e^{m_2} - e^{m_1}} = -C_1,\ A_1 = -\frac{\operatorname{Re} Gr}{e^{m_1} - e^{m_2}},$$

$$A_2 = \frac{A_1}{m_1(m_1 + \operatorname{Re})},\ A_3 = \frac{-A_1}{m_2(m_2 + \operatorname{Re})},\ C_4 = \frac{1}{1 - e^{-\operatorname{Re}}}\left[A_2(e^{m_1} - 1) + A_3(e^{m_2} - 1)\right],\ C_3 = -C_4 - A_2 - A_3,$$

$$X = \operatorname{Re}^2 + 4\pi^2,\ Y = -4\omega,\ G_1 = \frac{Y}{2G_2},\ G_2 = -\sqrt{\frac{X + \sqrt{X^2 + Y^2}}{2}},\ G_3 = \frac{\operatorname{Re} + G_1}{2},\ G_4 = \frac{G_2}{2},\ G_5 = \frac{\operatorname{Re} - G_1}{2},$$

$$G_6 = -\frac{G_2}{2},\ r_1 = \frac{\operatorname{Re} + \sqrt{\operatorname{Re}^2 + 4(\pi^2 - i\omega)}}{2} = G_3 + iG_4,\ r_2 = \frac{\operatorname{Re} - \sqrt{\operatorname{Re}^2 + 4(\pi^2 - i\omega)}}{2} = G_5 + iG_6,$$

$$G_7 = e^{-G_3}\cos G_4 - \frac{(\pi - G_3)}{2\pi}e^{\pi} - \frac{(\pi + G_3)}{2\pi}e^{-\pi},\ G_8 = -e^{-G_3}\sin G_4 + \frac{G_4}{2\pi}e^{\pi} - \frac{G_4}{2\pi}e^{-\pi},$$

$$G_9 = e^{-G_5}\cos G_6 - \frac{(\pi - G_5)}{2\pi}e^{\pi} - \frac{(\pi + G_5)}{2\pi}e^{-\pi},\ G_{10} = -e^{-G_5}\sin G_6 + \frac{G_6}{2\pi}e^{\pi} - \frac{G_6}{2\pi}e^{-\pi},$$

$$G_{11} = e^{-G_3}(G_3\cos G_4 + G_4\sin G_4) + \frac{(\pi - G_3)}{2}e^{\pi} - \frac{(\pi + G_3)}{2}e^{-\pi},$$

$$G_{12} = -e^{-G_3}(G_4\cos G_4 - G_3\sin G_4) - \frac{G_4}{2}e^{\pi} - \frac{G_4}{2}e^{-\pi},$$

$$G_{13} = e^{-G_5}(G_5\cos G_6 + G_6\sin G_6) + \frac{(\pi - G_5)}{2}e^{\pi} - \frac{(\pi + G_5)}{2}e^{-\pi},$$

$$G_{14} = -e^{-G_5}(G_6\cos G_6 - G_5\sin G_6) - \frac{G_6}{2}e^{\pi} - \frac{G_6}{2}e^{-\pi},\ r_3 = e^{-r_1} - \frac{(\pi - r_1)}{2\pi}e^{\pi} - \frac{(\pi + r_1)}{2\pi}e^{-\pi} = G_7 + iG_8,$$

$$r_4 = e^{-r_2} - \frac{(\pi - r_2)}{2\pi}e^{\pi} - \frac{(\pi + r_2)}{2\pi}e^{-\pi} = G_9 + iG_{10},\ r_5 = r_1 e^{-r_1} + \frac{(\pi - r_1)}{2}e^{\pi} - \frac{(\pi + r_1)}{2}e^{-\pi} = G_{11} + iG_{12},$$

$$r_6 = r_2 e^{-r_2} + \frac{(\pi - r_2)}{2}e^{\pi} - \frac{(\pi + r_2)}{2}e^{-\pi} = G_{13} + iG_{14},\ G_{15} = e^{\pi}(\pi G_9 + G_{13}) + e^{-\pi}(G_{13} - \pi G_9),$$

$$G_{16} = e^{\pi}(\pi G_{10} + G_{14}) + e^{-\pi}(G_{14} - \pi G_{10}),\ G_{17} = 2(G_7 G_{13} - G_8 G_{14} - G_9 G_{11} + G_{10} G_{12}),$$

$$G_{18} = 2(G_8 G_{13} + G_7 G_{14} - G_{10} G_{11} - G_9 G_{12}),\ G_{19} = \frac{G_{15}G_{17} + G_{16}G_{18}}{G_{17}^{\ 2} + G_{18}^{\ 2}},\ G_{20} = \frac{G_{16}G_{17} - G_{15}G_{18}}{G_{17}^{\ 2} + G_{18}^{\ 2}},$$

$$G_{21} = -e^{\pi}(\pi G_7 + G_{11}) - e^{-\pi}(G_{11} - \pi G_7),\ G_{22} = -e^{\pi}(\pi G_8 + G_{12}) - e^{-\pi}(G_{12} - \pi G_8),$$

$$G_{23}=\frac{G_{21}G_{17}+G_{22}G_{18}}{G_{17}{}^2+G_{18}{}^2},\ G_{24}=\frac{G_{22}G_{17}-G_{21}G_{18}}{G_{17}{}^2+G_{18}{}^2},\ G_{25}=-\frac{1}{2\pi}\{G_{19}(\pi-G_3)+G_4G_{20}+G_{23}(\pi-G_5)+G_6G_{24}+\pi\},$$

$$G_{26}=-\frac{1}{2\pi}\{G_{20}(\pi-G_3)-G_4G_{19}+G_{24}(\pi-G_5)-G_6G_{23}\},$$

$$G_{27}=-\frac{1}{2\pi}\{G_{19}(\pi+G_3)-G_4G_{20}+G_{23}(\pi+G_5)-G_6G_{24}+\pi\},$$

$$G_{28}=-\frac{1}{2\pi}\{G_{20}(\pi+G_3)+G_4G_{20}+G_{24}(\pi+G_5)+G_6G_{23}\},\ A=\frac{e^{\pi}(\pi r_4+r_6)+e^{-\pi}(r_6-\pi r_4)}{2(r_3r_6-r_4r_5)}=G_{19}+iG_{20},$$

$$B=\frac{-e^{\pi}(\pi r_3+r_5)-e^{-\pi}(r_5-\pi r_3)}{2(r_3r_6-r_4r_5)}=G_{23}+iG_{24},\ C=-\frac{1}{2\pi}[A(\pi-r_1)+B(\pi-r_2)+\pi]=G_{25}+iG_{26},$$

$$D=-\frac{1}{2\pi}[A(\pi+r_1)+B(\pi+r_2)+\pi]=G_{27}+iG_{28},\ X_1=\mathrm{Re}^2\,\mathrm{Pr}^2+4(\pi^2-Q),\ Y_1=-4\omega\,\mathrm{Pr},\ H_1=\frac{Y_1}{2H_2},$$

$$H_2=-\sqrt{\frac{X_1+\sqrt{X_1{}^2+Y_1^2}}{2}},\ H_3=\frac{-\mathrm{Re\,Pr}+H_1}{2},\ H_4=\frac{H_2}{2},\ H_5=\frac{-\mathrm{Re\,Pr}-H_1}{2},\ H_6=-\frac{H_2}{2},$$

$$r_7=\frac{-\mathrm{Re\,Pr}+\sqrt{\{\mathrm{Re}^2\,\mathrm{Pr}^2+4(\pi^2-Q)\}+i(-4\omega\,\mathrm{Pr})}}{2}=H_3+iH_4,$$

$$r_8=\frac{-\mathrm{Re\,Pr}-\sqrt{\{\mathrm{Re}^2\,\mathrm{Pr}^2+4(\pi^2-Q)\}+i(-4\omega\,\mathrm{Pr})}}{2}=H_5+iH_6,\ m_1-G_3-iG_4=H_7+iH_8,$$

$$m_1-G_5-iG_6=H_9+iH_{10},\ m_2-G_3-iG_4=J_7+iJ_8,\ m_2-G_5-iG_6=J_9+iJ_{10},$$

$$H_{15}=H_7{}^2-H_8{}^2+\mathrm{Re\,Pr}\,H_7-(\pi^2-Q),\ H_{16}=2H_7H_8+\mathrm{Re\,Pr}\,H_8+\omega\,\mathrm{Pr},\ H_{17}=\frac{G_{19}H_{15}+G_{20}H_{16}}{H_{15}{}^2+H_{16}{}^2},$$

$$H_{18}=\frac{G_{20}H_{15}-G_{19}H_{16}}{H_{15}{}^2+H_{16}{}^2},\ H_{19}=H_9{}^2-H_{10}{}^2+\mathrm{Re\,Pr}\,H_9-(\pi^2-Q),\ H_{20}=2H_9H_{10}+\mathrm{Re\,Pr}\,H_{10}+\omega\,\mathrm{Pr},$$

$$H_{21}=\frac{G_{23}H_{19}+G_{24}H_{20}}{H_{19}{}^2+H_{20}{}^2},\ H_{22}=\frac{G_{24}H_{19}-G_{23}H_{20}}{H_{19}{}^2+H_{20}{}^2},\ H_{23}=(\pi+m_1)^2+\mathrm{Re\,Pr}(\pi+m_1)-(\pi^2-Q),$$

$$H_{24}=H_{28}=\omega\,\mathrm{Pr},\ H_{25}=\frac{G_{25}H_{23}+G_{26}H_{24}}{H_{23}{}^2+H_{24}{}^2},\ H_{26}=\frac{G_{26}H_{23}-G_{25}H_{24}}{H_{23}{}^2+H_{24}{}^2},$$

$$H_{27}=(m_1-\pi)^2+\mathrm{Re\,Pr}(m_1-\pi)-(\pi^2-Q),\ H_{29}=\frac{G_{27}H_{27}+G_{28}H_{28}}{H_{27}{}^2+H_{28}{}^2},\ H_{30}=\frac{G_{28}H_{27}-G_{27}H_{28}}{H_{27}{}^2+H_{28}{}^2},$$

$$J_{15}=J_7{}^2-J_8{}^2+\mathrm{Re\,Pr}\,J_7-(\pi^2-Q),\ J_{16}=2J_7J_8+\mathrm{Re\,Pr}\,J_8+\omega\,\mathrm{Pr},\ J_{17}=\frac{G_{19}J_{15}+G_{20}J_{16}}{J_{15}{}^2+J_{16}{}^2},$$

$$J_{18}=\frac{G_{20}J_{15}-G_{19}J_{16}}{J_{15}{}^2+J_{16}{}^2},\ J_{19}=J_9{}^2-J_{10}{}^2+\mathrm{Re\,Pr}\,J_9-(\pi^2-Q),\ J_{20}=2J_9J_{10}+\mathrm{Re\,Pr}\,J_{10}+\omega\,\mathrm{Pr},$$

$$J_{21}=\frac{G_{23}J_{19}+G_{24}J_{20}}{J_{19}{}^2+J_{20}{}^2},\ J_{22}=\frac{G_{24}J_{19}-G_{23}J_{20}}{J_{19}{}^2+J_{20}{}^2},\ J_{23}=(\pi+m_2)^2+\mathrm{Re\,Pr}(\pi+m_2)-(\pi^2-Q),\ J_{24}=J_{28}=\omega\,\mathrm{Pr},$$

$$J_{25}=\frac{G_{25}J_{23}+G_{26}J_{24}}{J_{23}^{\ 2}+J_{24}^{\ 2}},\ J_{26}=\frac{G_{26}J_{23}-G_{25}J_{24}}{J_{23}^{\ 2}+J_{24}^{\ 2}},\ J_{27}=(m_2-\pi)^2+\operatorname{Re}\Pr(m_2-\pi)-(\pi^2-Q),$$

$$J_{29}=\frac{G_{27}J_{27}+G_{28}J_{28}}{J_{27}^{\ 2}+J_{28}^{\ 2}},\ J_{30}=\frac{G_{28}J_{27}-G_{27}J_{28}}{J_{27}^{\ 2}+J_{28}^{\ 2}},$$

$$H_{31}=\operatorname{Re}\Pr C_1m_1(H_{17}+H_{21}+H_{25}+H_{29})+\operatorname{Re}\Pr C_2m_2(J_{17}+J_{21}+J_{25}+J_{29}),$$

$$H_{32}=\operatorname{Re}\Pr C_1m_1(H_{18}+H_{22}+H_{26}+H_{30})+\operatorname{Re}\Pr C_2m_2(J_{18}+J_{22}+J_{26}+J_{30}),$$

$$H_{33}=\operatorname{Re}\Pr C_1m_1\{e^{H_7}(H_{17}\cos H_8-H_{18}\sin H_8)+e^{H_9}(H_{21}\cos H_{10}-H_{22}\sin H_{10})+e^{m_1+\pi}H_{25}+e^{m_1-\pi}H_{29}$$

$$+\operatorname{Re}\Pr C_2m_2\{e^{J_7}(J_{17}\cos J_8-J_{18}\sin J_8)+e^{J_9}(J_{21}\cos J_{10}-J_{22}\sin J_{10})+e^{m_2+\pi}J_{25}+e^{m_2-\pi}J_{29},$$

$$H_{34}=\operatorname{Re}\Pr C_1m_1\{e^{H_7}(H_{18}\cos H_8+H_{17}\sin H_8)+e^{H_9}(H_{22}\cos H_{10}+H_{21}\sin H_{10})+e^{m_1+\pi}H_{26}+e^{m_1-\pi}H_{30}$$

$$+\operatorname{Re}\Pr C_2m_2\{e^{J_7}(J_{18}\cos J_8+J_{17}\sin J_8)+e^{J_9}(J_{22}\cos J_{10}+J_{21}\sin J_{10})+e^{m_2+\pi}J_{26}+e^{m_2-\pi}J_{30},$$

$$H_{35}=e^{H_3}\cos H_4-e^{H_5}\cos H_6,\ H_{36}=e^{H_3}\sin H_4-e^{H_5}\sin H_6,$$

$$H_{37}=H_{33}-e^{H_3}H_{31}\cos H_4+e^{H_3}H_{32}\sin H_4,\ H_{38}=H_{34}-e^{H_3}H_{32}\cos H_4-e^{H_3}H_{31}\sin H_4,$$

$$H_{39}=\frac{H_{35}H_{37}+H_{36}H_{38}}{H_{35}^{\ 2}+H_{36}^{\ 2}},\ H_{40}=\frac{H_{35}H_{38}-H_{36}H_{37}}{H_{35}^{\ 2}+H_{36}^{\ 2}},\ H_{41}=-H_{31}-H_{39},\ H_{42}=-H_{32}-H_{40},$$

$$D_1=-\operatorname{Re}-G_3,\ D_2=-\operatorname{Re}-G_5,\ D_3=-\operatorname{Re}+\pi,\ D_4=-\operatorname{Re}-\pi,\ D_5=m_1-G_3,\ D_6=m_1-G_5,$$

$$D_7=m_1+\pi,\ D_8=m_1-\pi,\ D_9=m_2-G_3,\ D_{10}=m_2-G_5,\ D_{11}=m_2+\pi,\ D_{12}=m_2-\pi,$$

$$X_2=\operatorname{Re}^2+4\pi^2,\ Y_2=-4\omega,\ K_1=\frac{Y_2}{2K_2},\ K_2=-\sqrt{\frac{X_2+\sqrt{X_2^{\ 2}+Y_2^{\ 2}}}{2}},\ K_3=\frac{-\operatorname{Re}+K_1}{2},\ K_4=\frac{K_2}{2},\ K_5=\frac{-\operatorname{Re}-K_1}{2},$$

$$K_6=-\frac{K_2}{2},\ r_9=K_3+iK_4,\ r_{10}=K_5+iK_6,\ D_{13}=D_1^{\ 2}+G_4^{\ 2}+\operatorname{Re}D_1-\pi^2,\ D_{14}=-2D_1G_4-\operatorname{Re}G_4+\omega,$$

$$D_{15}=D_2^{\ 2}+G_6^{\ 2}+\operatorname{Re}D_2-\pi^2,\ D_{16}=-2D_2G_6-\operatorname{Re}G_6+\omega,\ D_{17}=D_3^{\ 2}+\operatorname{Re}D_3-\pi^2,\ D_{18}=D_4^{\ 2}+\operatorname{Re}D_4-\pi^2,$$

$$D_{19}=D_5^{\ 2}+G_4^{\ 2}+\operatorname{Re}D_5-\pi^2,\ D_{20}=-2D_5G_4-\operatorname{Re}G_4+\omega,\ D_{21}=D_6^{\ 2}+G_6^{\ 2}+\operatorname{Re}D_6-\pi^2,$$

$$D_{22}=-2D_6G_6-\operatorname{Re}G_6+\omega,\ D_{23}=D_7^{\ 2}+\operatorname{Re}D_7-\pi^2,\ D_{24}=D_8^{\ 2}+\operatorname{Re}D_8-\pi^2,\ D_{25}=D_9^{\ 2}+G_4^{\ 2}+\operatorname{Re}D_9-\pi^2,$$

$$D_{26}=-2D_9G_4-\operatorname{Re}G_4+\omega,\ D_{27}=D_{10}^{\ 2}+G_6^{\ 2}+\operatorname{Re}D_{10}-\pi^2,\ D_{28}=-2D_{10}G_6-\operatorname{Re}G_6+\omega,$$

$$D_{29}=D_{11}^{\ 2}+\operatorname{Re}D_{11}-\pi^2,\ D_{30}=D_{12}^{\ 2}+\operatorname{Re}D_{12}-\pi^2,\ D_{31}=H_3^{\ 2}-H_4^{\ 2}+\operatorname{Re}H_3-\pi^2,\ D_{32}=2H_3H_4+\operatorname{Re}H_4+\omega,$$

$$D_{33}=H_5^{\ 2}-H_6^{\ 2}+\operatorname{Re}H_5-\pi^2,\ D_{34}=2H_5H_6+\operatorname{Re}H_6+\omega,\ D_{35}=H_7^{\ 2}-H_8^{\ 2}+\operatorname{Re}H_7-\pi^2,$$

$$D_{36}=2H_7H_8+\operatorname{Re}H_8+\omega,\ D_{37}=H_9^{\ 2}-H_{10}^{\ 2}+\operatorname{Re}H_9-\pi^2,\ D_{38}=2H_9H_{10}+\operatorname{Re}H_{10}+\omega,$$

$$D_{39}=D_7^{\ 2}+\operatorname{Re}D_7-\pi^2,\ D_{40}=D_8^{\ 2}+\operatorname{Re}D_8-\pi^2,\ D_{41}=J_7^{\ 2}-J_8^{\ 2}+\operatorname{Re}J_7-\pi^2,\ D_{42}=2J_7J_8+\operatorname{Re}J_8+\omega,$$

$$D_{43}=J_9^{\ 2}-J_{10}^{\ 2}+\operatorname{Re}J_9-\pi^2,\ D_{44}=2J_9J_{10}+\operatorname{Re}J_{10}+\omega,\ D_{45}=D_{11}^{\ 2}+\operatorname{Re}D_{11}-\pi^2,\ D_{46}=D_{12}^{\ 2}+\operatorname{Re}D_{12}-\pi^2,$$

$$R=-\operatorname{Re}Gr,\ R_1=-\operatorname{Re}^2\Pr GrC_1m_1,\ R_2=-\operatorname{Re}^2\Pr GrC_2m_2,\ R_3=-\operatorname{Re}^2C_4G_{19},\ R_4=-\operatorname{Re}^2C_4G_{20},$$

$R_5 = -\mathrm{Re}^2 C_4 G_{23},\ R_6 = -\mathrm{Re}^2 C_4 G_{24},\ R_7 = -\mathrm{Re}^2 C_4 G_{25},$

$R_8 = -\mathrm{Re}^2 C_4 G_{26},\ R_9 = -\mathrm{Re}^2 C_4 G_{27},\ R_{10} = -\mathrm{Re}^2 C_4 G_{28},\ R_{11} = \mathrm{Re}\, m_1 A_2 G_{19},\ R_{12} = \mathrm{Re}\, m_1 A_2 G_{20},$

$R_{13} = \mathrm{Re}\, m_1 A_2 G_{23},\ R_{14} = \mathrm{Re}\, m_1 A_2 G_{24},\ R_{15} = \mathrm{Re}\, m_1 A_2 G_{25},\ R_{16} = \mathrm{Re}\, m_1 A_2 G_{26},\ R_{17} = \mathrm{Re}\, m_1 A_2 G_{27},$
$R_{18} = \mathrm{Re}\, m_1 A_2 G_{28},\ R_{19} = \mathrm{Re}\, m_2 A_3 G_{19},\ R_{20} = \mathrm{Re}\, m_2 A_3 G_{20},\ R_{21} = \mathrm{Re}\, m_2 A_3 G_{23},\ R_{22} = \mathrm{Re}\, m_2 A_3 G_{24},$

$R_{23} = \mathrm{Re}\, m_2 A_3 G_{25},\ R_{24} = \mathrm{Re}\, m_2 A_3 G_{26},\ R_{25} = \mathrm{Re}\, m_2 A_3 G_{27},\ R_{26} = \mathrm{Re}\, m_2 A_3 G_{28},$

$$D_{47} = \frac{R_3 D_{13} + R_4 D_{14}}{D_{13}^2 + D_{14}^2},\ D_{48} = \frac{R_4 D_{13} - R_3 D_{14}}{D_{13}^2 + D_{14}^2},\ D_{49} = \frac{R_5 D_{15} + R_6 D_{16}}{D_{15}^2 + D_{16}^2},\ D_{50} = \frac{R_6 D_{15} - R_5 D_{16}}{D_{15}^2 + D_{16}^2},$$

$$D_{51} = \frac{R_7 D_{17} + R_8 \omega}{D_{17}^2 + \omega^2},\ D_{52} = \frac{R_8 D_{17} - R_7 \omega}{D_{17}^2 + \omega^2},\ D_{53} = \frac{R_9 D_{18} + R_{10} \omega}{D_{18}^2 + \omega^2},\ D_{54} = \frac{R_{10} D_{18} - R_9 \omega}{D_{18}^2 + \omega^2},$$

$$D_{55} = \frac{R_{11} D_{19} + R_{12} D_{20}}{D_{19}^2 + D_{20}^2},\ D_{56} = \frac{R_{12} D_{19} - R_{11} D_{20}}{D_{19}^2 + D_{20}^2},\ D_{57} = \frac{R_{13} D_{21} + R_{14} D_{22}}{D_{21}^2 + D_{22}^2},\ D_{58} = \frac{R_{14} D_{21} - R_{13} D_{22}}{D_{21}^2 + D_{22}^2},$$

$$D_{59} = \frac{R_{15} D_{23} + R_{16} \omega}{D_{23}^2 + \omega^2},\ D_{60} = \frac{R_{16} D_{23} - R_{15} \omega}{D_{23}^2 + \omega^2},\ D_{61} = \frac{R_{17} D_{24} + R_{18} \omega}{D_{24}^2 + \omega^2},\ D_{62} = \frac{R_{18} D_{24} - R_{17} \omega}{D_{24}^2 + \omega^2},$$

$$D_{63} = \frac{R_{19} D_{25} + R_{20} D_{26}}{D_{25}^2 + D_{26}^2},\ D_{64} = \frac{R_{20} D_{25} - R_{19} D_{26}}{D_{25}^2 + D_{26}^2},\ D_{65} = \frac{R_{21} D_{27} + R_{22} D_{28}}{D_{27}^2 + D_{28}^2},\ D_{66} = \frac{R_{22} D_{27} - R_{21} D_{28}}{D_{27}^2 + D_{28}^2},$$

$$D_{67} = \frac{R_{23} D_{29} + R_{24} \omega}{D_{29}^2 + \omega^2},\ D_{68} = \frac{R_{24} D_{29} - R_{23} \omega}{D_{29}^2 + \omega^2},\ D_{69} = \frac{R_{25} D_{30} + R_{26} \omega}{D_{30}^2 + \omega^2},\ D_{70} = \frac{R_{26} D_{30} - R_{25} \omega}{D_{30}^2 + \omega^2},$$

$$D_{71} = R\left(\frac{H_{41} D_{31} + H_{42} D_{32}}{D_{31}^2 + D_{32}^2}\right),\ D_{72} = R\left(\frac{H_{42} D_{31} - H_{41} D_{32}}{D_{31}^2 + D_{32}^2}\right),\ D_{73} = R\left(\frac{H_{39} D_{33} + H_{40} D_{34}}{D_{33}^2 + D_{34}^2}\right),$$
$$D_{74} = R\left(\frac{H_{40} D_{33} - H_{39} D_{34}}{D_{33}^2 + D_{34}^2}\right),\ D_{75} = R_1\left(\frac{H_{17} D_{35} + H_{18} D_{36}}{D_{35}^2 + D_{36}^2}\right),\ D_{76} = R_1\left(\frac{H_{18} D_{35} - H_{17} D_{36}}{D_{35}^2 + D_{36}^2}\right),$$

$$D_{77} = R_1\left(\frac{H_{21} D_{37} + H_{22} D_{38}}{D_{37}^2 + D_{38}^2}\right),\ D_{78} = R_1\left(\frac{H_{22} D_{37} - H_{21} D_{38}}{D_{37}^2 + D_{38}^2}\right),\ D_{79} = R_1\left(\frac{H_{25} D_{39} + H_{26} \omega}{D_{39}^2 + \omega^2}\right),$$

$$D_{80} = R_1\left(\frac{H_{26} D_{39} - H_{25} \omega}{D_{39}^2 + \omega^2}\right),\ D_{81} = R_1\left(\frac{H_{29} D_{40} + H_{30} \omega}{D_{40}^2 + \omega^2}\right),\ D_{82} = R_1\left(\frac{H_{30} D_{40} - H_{29} \omega}{D_{40}^2 + \omega^2}\right),$$

$$D_{83} = R_2\left(\frac{J_{17} D_{41} + J_{18} D_{42}}{D_{41}^2 + D_{42}^2}\right),\ D_{84} = R_2\left(\frac{J_{18} D_{41} - J_{17} D_{42}}{D_{41}^2 + D_{42}^2}\right),\ D_{85} = R_2\left(\frac{J_{21} D_{43} + J_{22} D_{44}}{D_{43}^2 + D_{44}^2}\right),$$
$$D_{86} = R_2\left(\frac{J_{22} D_{43} - J_{21} D_{44}}{D_{43}^2 + D_{44}^2}\right),\ D_{87} = R_2\left(\frac{J_{25} D_{45} + J_{26} \omega}{D_{45}^2 + \omega^2}\right),\ D_{88} = R_2\left(\frac{J_{26} D_{45} - J_{25} \omega}{D_{45}^2 + \omega^2}\right),\ D_{89} = R_2\left(\frac{J_{29} D_{46} + J_{30} \omega}{D_{46}^2 + \omega^2}\right),$$
$$D_{90} = R_2\left(\frac{J_{30} D_{46} - J_{29} \omega}{D_{46}^2 + \omega^2}\right),\ D_{91} = D_{59} + D_{79},\ D_{92} = D_{60} + D_{80},\ D_{93} = D_{61} + D_{81},\ D_{94} = D_{62} + D_{82},$$

$D_{95} = D_{67} + D_{87}$, $D_{96} = D_{68} + D_{88}$, $D_{97} = D_{69} + D_{89}$, $D_{98} = D_{70} + D_{90}$,

$D_{99} = D_{47} + D_{49} + D_{51} + D_{53} + D_{55} + D_{57} + D_{63} + D_{65} + D_{71} + D_{73} + D_{75} + D_{77} + D_{83} + D_{85} + D_{91} + D_{93} + D_{95} + D_{97}$,

$D_{100} = D_{48} + D_{50} + D_{52} + D_{54} + D_{56} + D_{58} + D_{64} + D_{66} + D_{72} + D_{74} + D_{76} + D_{78} + D_{84} + D_{86} + D_{92} + D_{94} + D_{96} + D_{98}$,

$$D_{101} = e^{D_1}(D_{47}\cos G_4 + D_{48}\sin G_4) + e^{D_2}(D_{49}\cos G_6 + D_{50}\sin G_6) + e^{D_5}(D_{55}\cos G_4 + D_{56}\sin G_4)$$
$$+ e^{D_6}(D_{57}\cos G_6 + D_{58}\sin G_6) + e^{D_9}(D_{63}\cos G_4 + D_{64}\sin G_4) + e^{D_{10}}(D_{65}\cos G_6 + D_{66}\sin G_6)$$
$$+ e^{H_3}(D_{71}\cos H_4 - D_{72}\sin H_4) + e^{H_5}(D_{73}\cos H_6 - D_{74}\sin H_6) + e^{H_7}(D_{75}\cos H_8 - D_{76}\sin H_8)$$
$$+ e^{H_9}(D_{77}\cos H_{10} - D_{78}\sin H_{10}) + e^{J_7}(D_{83}\cos J_8 - D_{84}\sin J_8) + e^{J_9}(D_{85}\cos J_{10} - D_{86}\sin J_{10})$$
$$+ e^{D_3}D_{51} + e^{D_4}D_{53} + e^{D_7}D_{91} + e^{D_8}D_{93} + e^{D_{11}}D_{95} + e^{D_{12}}D_{97},$$

$$D_{102} = e^{D_1}(D_{48}\cos G_4 - D_{47}\sin G_4) + e^{D_2}(D_{50}\cos G_6 - D_{49}\sin G_6) + e^{D_5}(D_{56}\cos G_4 - D_{55}\sin G_4)$$
$$+ e^{D_6}(D_{58}\cos G_6 - D_{57}\sin G_6) + e^{D_9}(D_{64}\cos G_4 - D_{63}\sin G_4) + e^{D_{10}}(D_{66}\cos G_6 - D_{65}\sin G_6)$$
$$+ e^{H_3}(D_{72}\cos H_4 + D_{71}\sin H_4) + e^{H_5}(D_{74}\cos H_6 + D_{73}\sin H_6) + e^{H_7}(D_{76}\cos H_8 + D_{75}\sin H_8)$$
$$+ e^{H_9}(D_{78}\cos H_{10} + D_{77}\sin H_{10}) + e^{J_7}(D_{84}\cos J_8 + D_{83}\sin J_8) + e^{J_9}(D_{86}\cos J_{10} + D_{85}\sin J_{10})$$
$$+ e^{D_3}D_{52} + e^{D_4}D_{54} + e^{D_7}D_{92} + e^{D_8}D_{94} + e^{D_{11}}D_{96} + e^{D_{12}}D_{98},$$

$D_{103} = e^{K_3}\cos K_4 - e^{K_5}\cos K_6$, $D_{104} = e^{K_3}\sin K_4 - e^{K_5}\sin K_6$, $D_{105} = D_{101} - D_{99}e^{K_3}\cos K_4 + D_{100}e^{K_3}\sin K_4$,

$D_{106} = D_{102} - D_{99}e^{K_3}\sin K_4 - D_{100}e^{K_3}\cos K_4$,

$$D_{107} = \frac{D_{103}D_{105} + D_{104}D_{106}}{D_{103}^{\,2} + D_{104}^{\,2}},\ D_{108} = \frac{D_{103}D_{106} - D_{104}D_{105}}{D_{103}^{\,2} + D_{104}^{\,2}},\ D_{109} = -D_{99} - D_{107},\ D_{110} = -D_{100} - D_{108},$$

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