EDGE ODD GRACEFULNESS of the
GRAPH $P_m + 2S_n$, $K_{1,n}$ ($T_p$) $P_2$ and $S_3 \sqcup S_n$

C. VIMALA$^1$, S.CHITRA$^2$

$^1$Associate professor, Department of Mathematics, Periyar Maniyammai Institute of Science & Technology, Vallam, Thanjavur 613403, Tamilnadu, vimalasakthi@pmu.edu
$^2$M.Phil, Department of Mathematics, Periyar Maniyammai Institute of Science & Technology, Vallam, Thanjavur 613403, Tamilnadu, chitrafcp@gmail.com

Abstract: A (p, q) connected graph is edge odd graceful graph if there exists an injective map $f$: $E(G) \rightarrow \{1, 3, \ldots, (2q-1)\}$ so that induced map $f^+: V(G) \rightarrow \{0, 1, 2, \ldots, (2k-1)\}$ defined by $f^+_x \equiv \sum_{y \in N(x)} f(x, y) \pmod{2k}$, where the vertex $x$ is incident with other vertex $y$ and $k = \max\{p, q\}$ makes all the edges distinct and odd. Hence the graph $G$ is edge–odd graceful. In this article, the edge odd gracefulness of the graph $P_n + 2S_n$, $K_{1,n} \sqcup P_2 \sqcup S_3 \sqcup S_n$.

Keywords: Graceful Graphs, Odd graceful graph, Edge graceful, Edge odd graceful labeling, Edge odd graceful graph.

I. INTRODUCTION

A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. Graph labeling were first introduced in the mid 1960’s. In the intervening 50 years nearly 200 graph labeling techniques have been studied in over 2500 papers. Graph labeling have often been motivated by practical problem, is one of fascinating areas of research. A systematic study of various application of graph labeling carried out in Bloom and Golomb[1]. Labeled graph plays vital role to determine optimal circuit layout for computers and for the representation of compressed data structure. Let $G$ be a simple undirected graph with $q$ edges. Thought out this paper, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$, respectively. In 1967, Rosa [1] introduced a labeling of $G$ called graceful labeling which is an injection $f$ from $V(G)$ to the set $\{0, 1, 2, \ldots, q\}$ such that each edge $xy$ is assigned the label $|f(x)−f(y)|$, the resulting edge labels are distinct. A graph $G$ is said to be graceful if $G$ admits a graceful labeling.

In 1991, Gnanjothi [2] defined a graph $G$ to be odd-graceful if there is an injection $f$ from $V(G)$ to the set $\{0, 1, 2\ldots (2q − 1)\}$ such that, when each edge $xy$ is assigned the label $|f(x)−f(y)|$, the resulting edges labels are in the set $\{1, 3, 5\ldots, (2q −1)\}$. Later, Solairaju and Chithra [3] introduced a new type of labeling of a graph $G$ called an edge-odd graceful labeling which is a bijection $f$ from $E(G)$ to the set $\{1, 3, 5\ldots, 2q − 1\}$ so that the induced mapping $f^+$ from $V(G)$ to the set $\{0, 1, 2, \ldots, (2q −1)\}$ given by $f^+_x = P f(xy) \pmod{2q}$ where the vertex $x$ is adjacent to other vertex $y$. The edge labels and vertex labels are distinct. A graph that admitted an edge-odd graceful labeling is called edge-odd graceful.
Definition 1.1:
A graph G is an ordered triple set \((V(G), E(G), X(G))\) consisting of non-empty set V(G) of vertices, a set E(G) distinct from V(G) of edges and an incidence function X_G that associates with edge of G. If e is an edge and (u, v) are vertices such that X_G (e) = uv, then e is said to join the vertices u and v are called the ends of e.

Definition 1.2:
The book graph \(S_3 \square S_n\) (or Cartesian product of the star graphs \(S_3\) and \(S_n\)) is a connected graph obtained by adding 'n' number of \(C_4\) with one edge. It has 3n vertices and 5n-3 edges.

Definition 1.3:
The tensor product of two graphs \(G_1\) and \(G_2\) denoted by \(G_1 \times G_2\) has vertex set and the edge set \(V(G_1 \times G_2) = V(G_1) \times V\) and \(E(G_1 \times G_2) = \{(u_1, v_1) (u_2, v_2) / u_1, u_2 \in E(G_1) and v_1, v_2 \in E(G_2) \}\) respectively.

Definition 1.4:
\(P_m + 2S_n\) is a tree obtained from the path \(P_m\) by adding two star graph \(S_n\) to the each of the two pendant vertices. It has (2n+m) vertices and (2n+m-1) edges.

Definition 1.5:
A connected graph with p vertices and q edges is called graceful if it is possible to label the vertices with distinct integers \(f(x)\) from \(\{0,1,2,...,q\}\) in such a way that, when each edge xy is labelled with \(|f(x) - f(y)|\), the resulting edge labels are different.

Definition 1.6:
An edge graceful labelling on a simple graph on p vertices and q edges is a labelling of the edges by distinct integers in \(\{1,2,........,q\}\) such that the labelling on the vertices induced by labelling a vertex with sum of the incident edges taken mod p assigns all values from 0 to p-1 to the vertices.

Definition 1.7:
A graph \(G = (V(G), E(G))\) with p vertices and q edges is said to admit odd graceful labeling if \(f: V(G) \rightarrow \{0, 1, 2, ..., (2q - 1)\}\) is injective and the induced function \(f *: E(G) \rightarrow \{1, 3, ..., (2q - 1)\}\) defined as \(f *(e=uv) = |f(u) - f(v)|\) is bijective. A graph which admits odd graceful labeling is called an odd graceful graph.

Definition 1.8:
A \((p, q)\) connected graph is edge odd graceful graph if there exists an injective map \(f_+: E(G) \rightarrow \{1, 3, ..., (2q-1)\}\) so that induced map \(f_+: V(G) \rightarrow \{0, 1, 2, ..., (2k-1)\}\) defined by \(f_+(x) \equiv \sum f(x,y)(\text{mod } 2k)\), where the vertex x is incident with other vertex y and \(k=\max\{p, q\}\) makes all the edges distinct and odd.
II. MAIN RESULT

Theorem 2.1: The graph $P_m + 2S_n$, where $m \geq 5$ and $m, n$ is odd, is edge odd graceful graph.

Proof: Consider the graph $G = P_m + 2S_n$, it has $|V (G)| = (2n+m)$ vertices and $|E (G)| = (2n+m-1)$ edges. where $n$ is odd. The graph $P_m + 2S_n$, where $m, n$ is odd and $m \geq 5$ is shown in fig 1.

Define edge labeling $f: E (P_m + 2S_n) \rightarrow \{1, 3, ..., (2q - 1)\}$ as follows:

$f(e_i) = i, i = 1, 3, 5, ..., (m - 1)$

$f(e_i) = 2i - m, m \leq i \leq 2n + (m - 1)$

$f(e_i) = f(e_{2n+m-1}) + i, \quad i = 2, 4, ..., (m - 2)$

The above defined edge labeling function will induce the bijective vertex labeling function.

$f: V (G) \rightarrow \{0, 1, 2, ..., (2k - 1)\}$ such that $f_+ \equiv \{f(x, y)/ xy \in E mod 2k\}$ where the vertex x is incident with other vertex y and $k = \max \{p, q\}$ makes all the edges distinct and odd. Hence the graph admits the edge odd graceful labeling.

Theorem 2.2: The graph $K_{1,n} (T_p) P_2$, where $n$ is even, is an edge odd graceful graph.

Proof: Let $u_1, u_2, ..., u_{n+1}$ be the vertices of star $K_{1,n}$, with $u_1$ be the apex vertex. Let $v_1, v_2$ be the vertices of $P_2$. Let $G$ be the graph $K_{1,n} (T_p) P_2$. We divide the vertex of $G$ into two disjoint sets

$T_1 = \{(u_i, v_1) / i = 1, 2, ..., n\}$ and $T_2 = \{(u_i, v_2) / i = 1, 2, ..., n\}$
Define $f: E(G) \to \{1, 3, \ldots , 2q - 1\}$ as follows.

\[ f(u_i, V_1) = 2i - 1 , 1 \leq i \leq 2n \]
\[ f(u_i, V_2) = 2i - 1 , 2n < i \leq 4n \]

The above defined function $f$ provides graceful labeling for tensor product of $K_{1,n} (T_p) P_2$. That is, $K_{1,n} (T_p) P_2$ is an odd graceful graph.

**Theorem 2.3:** The connected graph $S_3 \Box S_n$, for $n \geq 3$, is edge – odd graceful.

**Proof:** The fig 3 is the Cartesian product graph $S_3 \Box S_n$ where $n$ is odd with $3n$ vertices and $5n-3$ edges, with some arbitrary labeling to its vertices and edges. It is proved that the graph $S_3 \Box S_n$, for $n \geq 3$, is edge – odd graceful.
Define \( f: E(G) \to \{1, 3, \ldots, (2q - 1)\} \) as follows \( f(e_i) = (2i - 1) \) for \( i = 1, 2, \ldots, (5n-3) \)

Then the induced vertex label \( f: V(G) \to \{0,1,2,\ldots,(2k - 1)\} \) such that \( f_+ \equiv \{f(x,y)/ xy \in E\} \mod 2k \), where this sum run over all edges through \( v \).

Hence the map \( f \) and the induced map \( f_+ \) provide labels as distinct odd numbers for edges and also the labelings for vertex set has distinct values in \( \{0, 1, 2, \ldots, (2k-1)\} \). Hence the graph, for \( n \) is odd, is edge-odd graceful.

**REFERENCES**


