Fixed Point Theorems and p-Lattice Ordered c-Distance for a Self Map in Ordered Cone Metric Spaces

Dr. Capt. K.Sujatha

Professor & Head

Department of Mathematics & Statistics

St. Joseph's College for Women (Autonomous)

Visakhapatnam-530 004, India

kambhampati.sujatha@gmail.com

Abstract:

Huang and Zhang [13] introduced the concept of the cone metric space, replacing the set of real numbers by an ordered Banach space, and they showed some fixed point theorems of contractive type mappings on cone metric spaces. Then, several fixed and common fixed point results in cone metric spaces were introduced in [2, 3, 9, 15, 24] and the references contained therein. The Banach contraction principle is the most celebrated fixed point theorem [6]. Afterward, some various definitions of contractive mappings were introduced by other researchers and several fixed and common fixed point theorems were considered in [7, 10, 17, 19, 25]. Also, the existence of fixed and common fixed points in partially ordered cone metric spaces was studied in [4, 5, 28]. In 1996, Kada et al. [18] defined the concept of *w*-distance in complete metric space and proved some fixed point theorems in complete metric spaces.

H. Rahimi, G. Soleimani Rad [22] extended the Banach contraction principle [6] and Chatterjea contraction theorem [7] on c-distance of Cho et al. [8], and proved some fixed point and common fixed point theorems in ordered cone metric spaces. In this paper we extend the results of [22]. Also we introduce the notion of p- lattice ordered c-distance and prove some fixed point theorems under a p-lattice ordered c-distance in ordered cone metric spaces, for a single function.

2 Preliminaries

First let us start with some basic definitions

Definition 2.1 ([13])

Let E be a real Banach space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{ 0 \}$

(ii) $ax + by \in P \ \forall \ x, y \in P$ and non-negative real numbers a and b.

(iii) $P \cap (-P) = \{0\}$.

Definition 2.2 ([13])

We define a partial ordering $\leq$ on E with respect to P and $P \subset E$ by $x \leq y$ if and only if $y - x \in P$. We shall write $x << y$ if $y - x \in \text{int } P$, int P denotes the interior of P. We denote by $\| . \|$ the norm on E. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies

$$\| x\| \leq K \| y\|$$

The least positive number K satisfying (1.14.1) is called the normal constant of P.

Definition 2.3 ([13])

A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{ x_n \}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \ldots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$

Definition 2.4 ([13])

Let X be a nonempty set and E be a real Banach space equipped with

the partial ordering $\leq$ with respect to the cone $P \subset E$. Suppose that the mapping

$d : X \times X \to E$ satisfies:

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X and (X,d) is called a cone metric space.

Definition 2.5 ([13])

Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x) \ll c$ for all $n > n_0$, and we write $\lim_{n\to\infty} d(x_n, x) = 0$.

(ii) $\{x_n\}$ is called a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists $n_0 \in N$ such that $d(x_n, x_m) \ll c$ for all $m,n > n_0$, and we write $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) If every Cauchy sequence in X is convergent, then X is called a complete cone metric space.

Lemma 2.6 ([13, 24])

Let (X,d) be a cone metric space and P be a normal cone with normal constant k. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y \in X$. Then the following hold:

(c1) $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.

(c2) If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$.

(c3) If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

(c4) If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.

(c5) $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n,m \to \infty$.

Lemma 2.7 ([4, 14])

Let E be a real Banach space with a cone P in E. Then, for all $u,v,w,c \in E$, the following hold:

(p1) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(p2) If $0 \leq u \ll c$ for each $c \in \text{int } P$, then $u = 0$.

(p3) If $u \leq \lambda u$ where $u \in P$ and $0 < \lambda < 1$, then $u = 0$.

(p4) Let $x_n \to 0$ in E, $0 \leq x_n$ and $0 \ll c$. Then there exists positive integer n_0 such that $x_n \ll c$ for each $n > n_0$.

(p5) If $0 \leq u \leq v$ and k is a nonnegative real number, then $0 \leq ku \leq kv$.

(p6) If $0 \leq u_n \leq v_n$ for all $n \in N$ and $u_n \to u$, $u_n \to u$ as $n \to \infty$, then $0 \leq u \leq v$.

Definition 2.8 ([8, 29])

Let (X,d) be a cone metric space. A function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied:

(q1) $0 \leq q(x, y)$ for all $x, y \in X$;

(q2) $q(x, z) \leq q(x, y) + q(y, z)$ for all $x, y, z \in X$;

(q3) for all $n \geq 1$ and $x \in X$, if $q(x, y_n) \leq u$ for some u , then $q(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Remark 2.9 ([8, 29])

Each *w*-distance q in a metric space (X,d) is a c-distance (with $E = R^+$ and $P = [0,\infty)$). But the converse does not hold. Therefore, the c-distance is a generalization of *w*-distance.

Examples 2.10 ([8, 28, 29])

(1) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(x, y)$ for all $x, y \in X$. Then q is a c-distance.

(2) Let $E = R$, $X = [0,\infty)$ and $P = \{x \in E : x \geq 0\}$. Define a mapping $d : X \times X \to E$ by

$d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping

$q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$. Then q is a c distance.

(3) Let $E = C^1{}_R[0,1]$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the cone

$P = \{x \in E : x(t) \geq 0$ on $[0,1]\}$. Also, let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by

$d(x, y) = |x-y|\psi$ for all $x, y \in X$, where $\psi : [0,1] \to R$ such that $\psi(t) = 2t$.

Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x, y) = (x + y)\psi$

for all $x, y \in X$. Then q is c-distance.

(4) Let (X,d) be a cone metric space and P be a normal cone. Put $q(x, y) = d(w, y)$

for all $x, y \in X$, where $w \in X$ is a fixed point. Then q is a c-distance.

Remark 2.11 ([8, 28, 29])

From Examples 2.10 [1,2,4], we have three important results

(i) Each cone metric d on X with a normal cone is a c-distance q on X.

(ii) For c-distance q, $q(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

(iii) For c-distance q, $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.

Lemma 2.12 ([8, 28, 29])

Let (X,d) be a cone metric space and let q be a c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$ are two sequences in P converging to 0. Then the following hold:

(qp1) If $q(x_n, y) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in N$, then $y = z$. Specifically,

if $q(x, y) = 0$ and $q(x, z) = 0$, then $y = z$.

(qp2) If $q(x_n, y_n) \preceq u_n$ and $q(x_n, z) \preceq v_n$ for $n \in N$, then $\{y_n\}$ converges to z.

(qp3) If $q(x_n, x_m) \preceq u_n$ for m > n, then $\{x_n\}$ is a Cauchy sequence in X.

(qp4) If $q(y, x_n) \preceq u_n$ for n ∈ N, then $\{x_n\}$ is a Cauchy sequence in X.

The following special case of (qp3) plays a crucial role in determining Cauchy sequences, in Section 3

Lemma 2.13 :

In addition to the hypothesis of Lemma 2.12, assume the following :

Let z ∈ P , $0 \le \mu < 1$ and $q(x_n, x_m) \le \mu^n z_0 \;\; \forall \; m \ge n.$

Then $\{ x_n \}$ is a Cauchy sequence. ($\because u_n = \mu^n z_0 \longrightarrow 0$ as $n \longrightarrow \infty$)

Definition 2.14 ([4, 8])

Let (X, ≤) be a partially ordered set. Two mappings *f*, *g* : X → X are said to be weakly increasing if $f x \le g f x$ and $g x \le f g x$ hold for all x ∈ X.

Definition 2.15 ([9])

A lattice is a partially ordered set S in which any two elements a,b ∈ S have the supremum (a ∪ b) and the infimum (a ∩ b). Sometimes we write max { a,b } for (a ∪ b) and min { a,b } for (a ∩ b).

H. Rahimi, G. Soleimani Rad [22] proved following theorem.

Theorem 2.16 : ([22] , Theorem 3.1)

Let (X, ≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let q be a c-distance on X and f : X → X be a continuous and non decreasing mapping with respect to ≤. Suppose that there exist mappings α,β,γ : X → [0,1) such that the following four conditions hold:

(i) $\alpha(fx) \le \alpha(x)$, $\beta(fx) \le \beta(x)$ and $\gamma(fx) \le \gamma(x)$ for all x ∈ X … (2.16.1)

(ii) $(\alpha + 2\beta + 2\gamma)(x) < 1$ for all $x \in X$ … (2.16.2)

(iii) for all $x,y \in X$ with $x \le y$, $q(fx,fy) \preceq \alpha(x)q(x,y) + \beta(x)q(x,fy) + \gamma(x)q(y,fx)$ … (2.16.3)

(iv) for all $x,y \in X$ with $x \le y$, $q(fy,fx) \preceq \alpha(x)q(y,x) + \beta(x)q(fy,x) + \gamma(x)q(fx,y)$ … (2.16.4)

If there exists $x_0 \in X$ such that $x_0 \le fx_0$, then f has a fixed point. Moreover, if $fz = z$,

then $q(z,z) = 0$.

Note : In the above theorem α, β, γ are functions of x. In section 3, we take α, β, γ to be constants so that above (i) becomes obvious.

3 Main results

In this section , we extend the result of H. Rahimi, G. Soleimani [22]. Also we introduce the notion of p-lattice ordered c-distance and prove some fixed point theorems under a p-lattice ordered c-distance in ordered cone metric spaces, for a single function.

Let us introduce the notion of a p-lattice ordered c-distance in cone metric spaces.

Definition : Suppose (X,d) is a cone metric space and $p : X \times X \to E$ is a c-distance. Clearly, the image $(X \times X)$ of $X \times X$ under p is a subset of P. If the image $p (X \times X)$ is a lattice in P, we say that p is a p-lattice ordered c-distance on X.

Now we state and prove two of our main results with the underlying space X having a p-lattice ordered c-distance. This becomes necessary since we consider maximum of three terms in the control function which we cannot do if the c-distance is not p-lattice ordered.

Theorem 3.1

Let $(X, \le)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let p be a p-lattice ordered c-distance on X and $f : X \to X$ be continuous non decreasing mapping with respect to $\le$. Suppose $\lambda \in [0, ½)$ and p satisfies the following conditions :

(i) $x \le y \le z$ implies $p(x,y) \le p(x,z)$ and $p(y,z) \le p(x,z)$ $\forall x,y,z \in X$ …(3.1.1)

(ii) $p(fx,fy) \le \lambda \max\{p(x,y), p(x,fy), p(y,fx)\}$ $if\ x \le y$ ….(3.1.2)

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$. Then f has a fixed point in X. Moreover, if $f z = z$, then $q(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f . Now,

suppose that $fx_0 \neq x_0$. Since f is non decreasing with respect to ≤ and $x_0 \leq fx_0$,

we get $f x_0 \leq f^2 x_0 \Longrightarrow x_1 \leq x_2$

In a similar way we can show that $x_n \leq x_{n+1}$ ∀ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_n$ and $y = x_{n-1}$ in (3.1.2), we have

$\therefore\ p\,(x_n, x_{n+1})$

$= p\,(f x_{n-1}, f x_n)$

$\leq \lambda\, max\,\{\, p\,(x_{n-1}, x_n), p\,(x_{n-1}, f x_n), p(x_n, f x_{n-1})\}$ ($\because x_{n-1} \leq x_n$)

$= \lambda\, max\,\{\, p\,(x_{n-1}, x_n), p\,(x_{n-1}, x_{n+1}), p\,(x_n, x_n)\}$

$= \lambda\ p\,(x_{n-1}, x_{n+1})$ (from (3.1.1))

$\leq \lambda\,\{\, p(x_{n-1}, x_n) + p\,(x_n, x_{n+1})\,\}$

$\Longrightarrow p\,(x_n, x_{n+1}) \leq (\frac{\lambda}{1-\lambda})\, p\,(x_{n-1}, x_n)$

∴ By induction ,

$$p\,(x_n, x_{n+1}) \leq (\frac{\lambda}{1-\lambda})^{n-1}\, p(x_0\,, x_1) \quad \forall\, n \in N$$

Now $\frac{\lambda}{1-\lambda} < 1$ *since* $0 \leq \lambda < ½$

∴ By taking $\mu = \frac{\lambda}{1-\lambda}$ and $z_0 = p\,(x_0\,, x_1) \in P$, from Lemma 2.13, we get that $\{\, x_n \,\}$ is a

Cauchy sequence in X. Since X is complete, there exists a point z ∈ X such that $x_n \to z$ as n → ∞ . Continuity of f implies that $x_{n+1} = fx_n \to fz$ as n → ∞ and since the limit of a sequence is unique, we get that $fz = z$. Thus, z is a fixed point of f.

since $fz = z$, $p(z,z) = p(fz, fz)$

$$\leq \lambda \max \{ p(z,z), p(z,fz), p(z,fz)\}$$

$$= \lambda \max \{ p(z,z), p(z,z), p(z,z)\}$$

$$= \lambda p(z,z)$$

$$\Longrightarrow p(z,z) = 0$$

Theorem 3.2 : Let (X , ≤) be a partially ordered set and (X, d) be a complete cone metric space. Also, let p be a p-lattice ordered c-distance on X and f : X → X be continuous non decreasing mapping with respect to ≤ . Suppose $\lambda \in [0, ½)$ and p satisfies the following conditions :

(i) $x \leq y \leq z$ implies $p(y,x) \leq p(z,x)$ and $p(z,y) \leq p(z,x)$ $\forall\, x,y,z \in X$ …(3.2.1)

(ii) $p(fy,fx) \leq \lambda \max \{p(y,x), p(fy,x), p(fx,y)\}$ $if\ x \leq y$ ….(3.2.2)

Suppose there exists $x_0 \in X$ such that $x_0 \leq f x_0$. Then f has a fixed point in X.

Moreover, if $fz = z$, then $p(z,z) = 0$.

Proof : As in Theorem 3.1 , write $x_n = f x_{n-1}$, n = 1,2,3,…

Then $x_n \leq x_{n+1}$ for n = 0,1,2,… since f is non- decreasing

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_{n-1}$ and $y = x_n$ in (3.1.2), we have

∴ $p(x_{n+1}, x_n)$

$= p(fx_n, fx_{n-1})$

$\leq \lambda \, max \, \{ \, p \, (x_n, x_{n-1}), p \, (fx_n, x_{n-1}), p \, (fx_{n-1}, x_n)\}$ (∵ by (3.2.1))

$= \lambda \, max \, \{ \, p \, (x_n, x_{n-1}), p \, (x_{n+1}, x_{n-1}), p \, (x_n, x_n)\}$

$= \lambda \; p \, (x_{n+1}, x_{n-1})$ (from 3. 2.1)

$\leq \lambda \, \{ \, p \, (x_{n+1}, x_n) \; + \; p \, (x_n, x_{n-1}) \, \}$

$\Longrightarrow p \, (x_{n+1}, x_n) \;\; \leq (\frac{\lambda}{1-\lambda}) \, p \, (x_n, x_{n-1})$

∴ By induction ,

$p \, (x_{n+1}, x_n) \;\; \leq (\frac{\lambda}{1-\lambda})^{n-1} \, p \, (x_1 \, , x_0)$ for n = 1,2,3,…

Now $\frac{\lambda}{1-\lambda} < 1$ since $0 \leq \lambda < ½$

∴ By Lemma 2.13 , $\{ \, x_n \, \}$ is a Cauchy sequence in X. Since X is complete, there exists a point z ∈ X such that $x_n \rightarrow z$ as n → ∞ . Continuity of f implies that $x_{n+1} = fx_n \rightarrow fz$ as n → ∞ so that $fz = z$. Thus, z is a fixed point of f.

since $fz = z$, $p \, (z, z) \;\; = p \, (fz, fz)$

$\leq \; \lambda \, max \, \{ \, p \, (z, z), p \, (fz, z), p \, (fz, z)\}$

$= \; \lambda \, max \, \{ \, p \, (z, z), p \, (z, z), p \, (z, z)\}$

$= \; \lambda \, p \, (z, z)$

$\Longrightarrow \; p \, (z, z) = 0.$

In the following theorem we obtain a condition under which a function may admit unique fixed point.

Theorem 3.3 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let p be a p-lattice ordered c-distance on X and f : X → X be continuous non decreasing mapping with respect to ≤ . Suppose $\lambda \in [\, 0, ½ \,)$ and p satisfies the following conditions :

(i) $x \le y \le z$ implies $p(x,y) \le p(x,z)$; $p(y,x) \le p(z,x)$ …(3.3.1)

$p(y,z) \le p(x,z)$; $p(z,y) \le p(z,x)$ $\forall\ x,y,z \in X$ … (3.3.2)

(ii) $p(fx,fy) \le \lambda max\{p(x,y), p(x,fy), p(y,fx)\}$ $if\ x \le y$ … (3.3.3)

(iii) $p(fy,fx) \le \lambda max\{p(y,x), p(fy,x), p(fx,y)\}$ $if\ x \le y$ ...(3.3.4)

Suppose there exists $x_0 \in X$ such that $x_0 \le f x_0$. Then f has a fixed point in X and no two fixed points are comparable. Moreover, if $fz = z$, then $p(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f . Now, suppose that $f x_0 \ne x_0$. Since f is non decreasing with respect to $\le$ and $x_0 \le fx_0$,

we get $f x_0 \le f^2 x_0 \Longrightarrow x_1 \le x_2$

In a similar way we can show that $x_n \le x_{n+1}$ $\forall$ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x = x_n$ and $y = x_{n-1}$ in (3.3.3), we have

$\therefore\ p\ (x_n, x_{n+1})$

$= p\ (f x_{n-1}, f x_n)$

$\le \lambda\ max\ \{\ p\ (x_{n-1}, x_n), p\ (x_{n-1}, f x_n), p\ (x_n, f x_{n-1})\}$ $(\because x_{n-1} \le x_n)$

$= \lambda\ max\ \{\ p\ (x_{n-1}, x_n), p\ (x_{n-1}, x_{n+1}), p\ (x_n, x_n)\}$

$= \lambda\ p\ (x_{n-1}, x_{n+1})$ (from 3.3.1)

$\le \lambda\ \{\ p\ (x_{n-1}, x_n) + p\ (x_n, x_{n+1})\ \}$

$\Longrightarrow p\ (x_n, x_{n+1}) \le (\frac{\lambda}{1-\lambda})\ p\ (x_{n-1}, x_n)$

$\therefore$ By induction ,

$$p\,(x_n, x_{n+1}) \;\leq (\tfrac{\lambda}{1-\lambda})^{n-1}\, p\,(x_0\,, x_1)$$

Similarly, using (3.3.4), we get

$p\,(x_{n+1}, x_n)$

$= \; p\,(f x_n, f x_{n-1})$

$\leq \lambda\, max\,\{\, p\,(x_n, x_{n-1}), p\,(f x_n, x_{n-1}), p\,(f x_{n-1}, x_n)\}$ (∵ by (3.2.1))

$= \; \lambda\, max\,\{\, p\,(x_n, x_{n-1}), p\,(x_{n+1}, x_{n-1}), p\,(x_n, x_n)\}$

$= \; \lambda \; p\,(x_{n+1}, x_{n-1})$ (from 3. 2.1)

$\leq \; \lambda\,\{\, p\,(x_{n+1}, x_n) + p\,(x_n, x_{n-1})\,\}$

$\Longrightarrow p\,(x_{n+1}, x_n) \;\; \leq (\tfrac{\lambda}{1-\lambda})\, p\,(x_n, x_{n-1})$

∴ By induction ,

$$p\,(x_{n+1}, x_n) \;\leq (\tfrac{\lambda}{1-\lambda})^{n-1}\, p\,(x_1\,, x_0) \quad \text{for } n = 1,2,3,\ldots$$

Now $\frac{\lambda}{1-\lambda} < 1 \;\; since \;\; 0 \leq \lambda < ½$

∴ By Lemma 2.11 , $\{\, x_n \,\}$ is a Cauchy sequence in X. Since X is complete, there exists a point z ∈ X such that $x_n \to z$ as n → ∞ .

∴ By continuity of f we get

$x_{n+1} = f x_n \to f z$ as n → ∞ .

Hence $fz = z$. Thus, z is a fixed point of f.

Now $\quad p\,(z, z) \;= p\,(f z, f z)$

$\leq \; \lambda\, max\,\{\, p\,(z, z), p\,(z, f z), p\,(z, f z)\}$

$= \lambda\, max\,\{\, p\,(z, z), p\,(z, z), p\,(z, z)\}$

$= \lambda\, p\,(z,z)$

$\Rightarrow\; p\,(z,z) = 0$

Uniqueness : Suppose z and z′ are two comparable fixed points of f so that $fz = z$ and

$fz' = z'$

We may suppose without loss of generality that $z \leq z'$

Now $\quad p\,(z,z')$

$= p\,(fz, fz')$

$\leq \lambda\, max\,\{\, p\,(z,z'), p\,(z,fz'), p\,(z',fz)\}$ (since $z \leq z'$ from (3.3.3))

$= \lambda\, max\,\{\, p\,(z,z'), p\,(z,z'), p\,(z',z)\}$

$= \lambda\, max\{p\,(z,z'), p\,(z',z)\}$

$\therefore\; p\,(z,z') \;\leq\; \lambda\, max\{p\,(z,z'), p\,(z',z)\}$ …(3.3.5)

Also $p\,(z',z) \; = p\,(fz', fz)$

$\leq \lambda\, max\,\{\, p\,(z',z), p\,(fz',z), p\,(fz,z')\}$ (since $z \leq z'$ from (3.3.4))

$= \lambda\, max\,\{\, p\,(z',z), p\,(z',z), p\,(z,z')\}$

$= \lambda\, max\,\{\, p\,(z',z), p\,(z,z')\}$

$\therefore\; p\,(z',z) \;\leq\; \lambda\, max\{p\,(z',z), p\,(z,z')\}$ …(3.3.6)

Suppose $\max\{p\,(z,z'), p\,(z',z)\} \;\neq\; 0$

Then from (3.3.5) and (3.3.6)

$\max\{p\,(z,z'), p\,(z',z)\} \leq \lambda\, max\,\{p\,(z',z), p\,(z,z')\}$

$< \; max\,\{p\,(z',z), p\,(z,z')\}$, a contradiction

Hence $\max\{p\,(z',z), p\,(z,z')\} = 0$

Consequently, $p\,(z', z) = 0 = p\,(z, z')$

$\therefore \quad z = z'$

Hence f cannot have two comparable fixed points.

The following theorem which is an analogue of theorem 3.1, for decreasing functions can be easily established.

Theorem 3.4 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let p be a p-lattice ordered c-distance on X and $f : X \to X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [0, ½)$ and p satisfies the following conditions :

(i) $x \leq y \leq z$ implies $p(x, y) \leq p(x, z)$ and $p(y, z) \leq p(x, z)$ $\forall x, y, z \in X$ …(3.4.1)

(ii) $p(fx, fy) \leq \lambda \max\{p(x, y), p(x, fy), p(fx, y)\}$ $if\ x \leq y$ ….(3.4.2)

Suppose there exists $x_0 \in X$ such that $x_0 \geq f x_0$. Then f has a fixed point in X. Moreover and if $f z = z$, then $p(z,z) = 0$.

The following theorem which is an analogue of theorem 3.2, for decreasing functions can be easily established.

Theorem 3.5 : Let $(X, \leq)$ be a partially ordered set and (X, d) be a complete cone metric space. Also, let p be a p-lattice ordered c-distance on X and $f : X \to X$ be continuous non decreasing mapping with respect to $\leq$. Suppose $\lambda \in [0, ½)$ and p satisfies the following conditions :

(i) $x \leq y \leq z$ implies $p(y, x) \leq p(z, x)$ and $p(z, y) \leq p(z, x)$ $\forall x, y, z \in X$ …(3.5.1)

(ii) $p(fy, fx) \leq \lambda \max\{p(y, x), p(fy, x), p(y, fx)\}$ $if\ x \leq y$ ….(3.5.2)

Suppose there exists $x_0 \in X$ such that $x_0 \geq f x_0$. Then f has a fixed point in X. Moreover, if $f z = z$, then $p(z,z) = 0$.

Now we prove an improved version of theorem 2.16

Theorem 3.6 : Let (X,≤) be a partially ordered set and (X,d) be a complete cone metric space. Also, let p be a c-distance on X and $f: X \rightarrow X$ be a continuous and non decreasing mapping with respect to ≤. Suppose $x_0 \in X$ such that $x_0 \leq f x_0$. Write $x_n = f^n x_0$,

$n = 0,1,2, \dots$ Suppose there exist α,β,γ with $\alpha + 2\beta + 2\gamma < 1$ (non negative constants) such that

$$p(x_{n+1}, x_{m+1}) \leq \alpha p(x_n, x_m) + \beta p(x_n, x_{m+1}) + \gamma p(x_m, x_{n+1}) \quad \dots(3.6.1)$$

$$p(x_{m+1}, x_{n+1}) \leq \alpha p(x_m, x_n) + \beta p(x_{m+1}, x_n) + \gamma p(x_{n+1}, x_m) \quad \dots(3.6.2)$$

for n = 0,1,2,… and m > n.

Then $\{x_n\}$ is a Cauchy sequence with limit z (say) and z is a fixed point of f.

Moreover, if $fz = z$, then $p(z,z) = 0$.

Proof : Write $x_n = f x_{n-1}$, n = 1,2,3,…

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = f x_n$ so that x_n is a fixed point of f .Now,

suppose that $fx_0 \neq x_0$. Since f is non decreasing with respect to ≤ and $x_0 \leq fx_0$,

we get $f x_0 \leq f^2 x_0 \Longrightarrow x_1 \leq x_2$

In a similar way we can show that $x_n \leq x_{n+1}$ ∀ n = 0,1,2,…

Here $x_n = f x_{n-1} = f^n x_0$

Now, let $x_n = x_n$ and $x_m = x_{n+1}$ in (3.6.1), we have

$$p(x_{n+1}, x_{n+2}) = p(f x_n, f x_{n+1})$$

$$\leq \alpha p(x_n, x_{n+1}) + \beta p(x_n, x_{n+2}) + \gamma p(x_{n+1}, x_{n+1})$$

$$\leq \alpha p(x_n, x_{n+1}) + \beta\{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})\} + \gamma\{p(x_{n+1}, x_n) + p(x_n, x_{n+1})\}$$

$$= (\alpha + \beta + \gamma)\, p(x_n, x_{n+1}) + \beta\, p(x_{n+1}, x_{n+2}) + \gamma\, p(x_{n+1}, x_n)$$

$\therefore p(x_{n+1}, x_{n+2}) \leq (\alpha+\beta+\gamma)\, p(x_n, x_{n+1}) + \beta\, p(x_{n+1}, x_{n+2}) + \gamma\, p(x_{n+1}, x_n)$ …(3.6.3)

Now $p(x_{n+2}, x_{n+1})$

$= p(fx_{n+1}, fx_n)$

$\leq \alpha p(x_{n+1}, x_n) + \beta p(x_{n+2}, x_n) + \gamma p(x_{n+1}, x_{n+1})$ (from (3.6.2))

$\leq \alpha p(x_{n+1}, x_n) + \beta\{p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)\} +$

$\gamma\{p(x_{n+1}, x_n) + p(x_n, x_{n+1})\}$

$= (\alpha+\beta+\gamma)\, p(x_{n+1}, x_n) + \beta\, p(x_{n+2}, x_{n+1}) + \gamma\, p(x_n, x_{n+1})$

$\therefore p(x_{n+2}, x_{n+1}) \leq (\alpha+\beta+\gamma)\, p(x_{n+1}, x_n) + \beta\, p(x_{n+2}, x_{n+1}) + \gamma\, p(x_n, x_{n+1})$ ...(3.6.4)

Combining (3.6.3) and (3.6.4) we get

$p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+1}) \leq (\alpha+\beta+\gamma)\{p(x_n, x_{n+1}) + p(x_{n+1}, x_n)\}$

$+\beta\,\{p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+1}) + \gamma\{ p(x_{n+1}, x_n) + p(x_n, x_{n+1})\}$

$\Longrightarrow \lambda_n \leq (\alpha+\beta+\gamma)\, \lambda_{n-1} + \beta\, \lambda_n + \gamma\, \lambda_{n-1}$ where

$\lambda_n = p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+1})$

$\Longrightarrow \quad \lambda_n \leq \left(\frac{\alpha+\beta+2\gamma}{1-\beta}\right) \lambda_{n-1}$

$\therefore \quad \lambda_n \leq \mu\ \lambda_{n-1}$, n = 1,2,3,… where $\mu = \frac{\alpha+2\beta+2\gamma}{1-\beta} < 1$

$\therefore$ By Lemma 2.11, $\{ x_n \}$ is a Cauchy sequence in X. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Continuity of f implies that $x_{n+1} = fx_n \to fz$ as $n \to \infty$ so that $fz = z$. Thus, z is a fixed point of f.

Since $fz = z$, $p(z, z) = p(fz, fz)$

$\leq \alpha p(z, z) + \beta p(z, z) + \gamma p(z, z)$

$$= (\alpha + \beta + \gamma)\, p(z, z)$$

$$< p(z, z)\ (\alpha + \beta + \gamma < \alpha + 2\beta + 2\gamma < 1)$$

$$\Longrightarrow p\,(z, z) = 0.$$

Now we show that theorem 2.16 is a simple consequence of theorem 3.6.

References

[1] ***M. Abbas, D. Ili´c, M. Ali Khan****, Coupled coincidence point and coupled common fixed point theorems in partially ordered metric spaces with w-distance, Fixed Point Theory and Applications 11 (2010) Article ID 134897.*

[2] ***M. Abbas, G. Jungck****, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, Journal of Mathematical Analysis and Applications 341 (2008) 416-420.*

[3] ***M. Abbas, B. E. Rhoades, T. Nazir****, Common fixed points for four maps in cone metric spaces, Applied Mathematics and Computation 216 (2010) 80-86.*

[4] ***I. Altun, B. Damnjanovi´c, D. Djori´c,*** *Fixed point and common fixed point theorems on ordered cone metric spaces, Applied Mathematics Letters 23 (2010) 310-316.*

[5] ***I. Altun, G. Durmaz****, Some fixed point theorems on ordered cone metric spaces, Rendiconti del Circolo Matematico di Palermo 58 (2009) 319-325.*

[6] ***S. Banach****, Sur les oprations dans les ensembles abstraits et leur application aux equations intgrales, Fundamenta Mathematicae 3 (1922) 133-181.*

[7] ***S. K. Chatterjea****, Fixed point theorems, Comptes rendus de l'Academie bulgare des Sciences 25 (1972) 727-730.*

[8] ***Y. J. Cho, R. Saadati, S. H. Wang****, Common fixed point theorems on generalized distance in ordered cone metric spaces, Computers and Mathematics with Applications 61 (2011) 1254-1260.*

[9] ***B. S. Choudhury, N. Metiya****, The point of coincidence and common fixed point for a pair of mappings in cone metric spaces, Computers and Mathematics with Applications 60 (2010) 1686-1695.*

[10] ***L. B. ´Ciri´c,*** *A generalization of Banach's contraction principle, Proceedings of the American Mathematical Society 45 (1974) 267-273.*

[11] ***K. Deimling****, Nonlinear Functional Analysis, Springer-Verlag, (1985).*

[12] ***M. Dordevi´c, D. Dori´c, Z. Kadelburg, S. Radenovi´c, D. Spasi´c,*** *Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory and Applications (2011) http://dx.doi.org/10.1186/ 1687-1812-2011-29/.*

[13] ***L. G. Huang, X. Zhang,*** *Cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 332 (2007) 1467-1475.*

[14] ***D. Ili´c, V. Rakoˇcevi´c,*** *Quasi contraction on a cone metric space, Applied Mathematics Letters 22 (2009) 728-731.*

[15] ***S. Jankovi´c, Z. Kadelburg, S. Radenovi´c****, On cone metric spaces, a survey, Nonlinear H. Rahimi, et al /IJIM Vol. 6, No. 2 (2014) 97-105 105*

Analysis, Theory, Methods & Applications 74 (2011) 2591-2601.

[16] ***G. Jungck,*** *Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East Journal of Mathematical Sciences 4 (1996) 199-215.*

[17] ***G. Jungck****, Commuting maps and fixed points, American Mathematic Monthly 83 (1976) 261-263.*

[18] ***O. Kada, T. Suzuki, W. Takahashi****, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Mathematica Japonica 44 (1996) 381-391.*

[19] ***R. Kannan****, Some results on fixed points, Bulletin of Calcutta Mathematical Society 60 (1968) 71-76.*

[20] ***H. Lakzian, F. Arabyani****, Some fixed point theorems in cone metric spaces with wdistance, International Journal of Mathematical Analysis 3 (2009) 1081-1086.*

[21] ***S. K. Mohanta****, Generalized w-distance and a fixed point theorem, International Journal of Contemporary Mathematical Sciences 6 (2011) 853-860.*

[22] ***H. Rahimi , G. Soleimani Rad ,*** *Fixed point theorems under c-distance in ordered cone metric space, Int. J. Industrial Mathematics (ISSN 2008-5621),Vol. 6, No. 2, 2014 Article ID IJIM-00253, 9 pages*

[23] ***A. Razani, Z. M. Nezhad, M. Boujary****, A fixed point theorem for w-distance, Applied Science 11 (2009) 114-117.*

[24] ***S. Rezapour, R. Hamlbarani****, Some note on the paper cone metric spaces and fixed point theorems of contractive mappings, Journal of Mathematical Analysis and Applications 345 (2008) 719-724.*

[25] ***B. E. Rhoades****, A comparison of various definition of contractive mappings, Transactions of the American Mathematical Society 266 (1977) 257-290.*

[26] ***R. Saadati, D. ORegan, S. M. Vaezpour, J. K. Kim,*** *Generalized distance and common fixed point theorems in Menger probabilistic metric spaces, Bulletin of the Iranian Mathematical Society 35 (2009) 97-117.*

[27] ***W. Shatanawi****, Partially ordered cone metric spaces and coupled fixed point results, Computers and Mathematics with Applications 60 (2010) 2508-2515.*

[28] ***W. Sintunavarat, Y. J. Cho, P. Kumam****, Common fixed point theorems for c-distance in ordered cone metric spaces, Computers and Mathematics with Applications 62 (2011) 1969-1978.*

[29] ***S. Wang, B. Guo,*** *Distance in cone metric spaces and common fixed point theorems, Applied Mathematics Letters 24 (2011) 17351739.*